Proof that any linear combination of degenerate eigenfunction is itself an eigenfunction

for doubly-degenerate set of eigenfns

given

$\hat{A}f = af$ and $\hat{A}g = ag$ & $\hat{A}$ is a linear operator
a is a constant

show $\hat{A}(f+g) = a(f+g)$

$\hat{A}(f+g) = \hat{A}(f+g)$ since $\hat{A}$ is linear

$= \hat{A}f + \hat{A}g$ since f & g are eigenfns of $\hat{A}$

$= af + ag$ pulling out constant a

$A(f+g) = a(f+g)$ Q.E.D.

general case (n-degenerate)

given $\hat{A}f_i = af_i$ $i = 1, 2, 3 \cdots n$ $\hat{A}$ is linear operator

show $\hat{A}\left(\sum_{i=1}^{n} f_i\right) = a\left(\sum_{i=1}^{n} f_i\right)$

$\hat{A}\left(\sum_{i=1}^{n} f_i\right) = \sum_{i=1}^{n} \hat{A} f_i$ since $\hat{A}$ is linear

$= \sum_{i=1}^{n} a f_i$ since all f_i are eigenfns of $\hat{A}$

$= a\left(\sum_{i=1}^{n} f_i\right)$

Q.E.D.