Vector Space Examples
Math 203
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Example:

\[ S = \text{set of all doubly infinite sequences of real numbers} = \{\{y_k\} : k \in \mathbb{Z}, y_k \in \mathbb{R}\} \]

Define \( \{y_k\} + \{z_k\} = \{y_k + z_k\} \) and \( c\{y_k\} = \{cy_k\} \).

Example: \( y_k = 2^k, \ z_k = \cos(k\pi/2), c = -2 \)

\[
\{y_k\} = (\ldots, 1, \frac{1}{2}, 1, 2, 4, \ldots) \\
\{z_k\} = (\ldots, \cos(-\pi), \cos(-\pi/2), \cos(0), \cos(\pi/2), \cos(\pi), \ldots) = (\ldots, -1, 0, 1, 0, -1, \ldots) \\
\{y_k\} + \{z_k\} = (\ldots, -\frac{3}{2}, \frac{1}{2}, 2, 2, 3, \ldots) \\
\frac{1}{3}\{y_k\} = (\ldots, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots)
\]

Claim: The set \( S \) of all doubly infinite sequences of real numbers is a vector space.

Proof: Let \( \{u_k\}, \{v_k\}, \) and \( \{w_k\} \) be any three doubly infinite sequences in \( S \), and let \( c \) and \( d \) be any two scalars.

1. \( \{u_k\} + \{v_k\} \) is in \( S \) since \( \{u_k + v_k\} \) is a doubly infinite sequence.

2. \( \{u_k\} + \{v_k\} = \{v_k\} + \{u_k\} \) because:

\[
\{u_k\} + \{v_k\} = \{u_k + v_k\} \quad \text{definition of addition in } S \\
= \{v_k + u_k\} \quad \text{addition is commutative in } \mathbb{R} \\
= \{v_k\} + \{u_k\} \quad \text{definition of addition in } S
\]

3. \( (\{u_k\} + \{v_k\}) + \{w_k\} = \{u_k\} + (\{v_k\} + \{w_k\}) \) because:

\[
(\{u_k\} + \{v_k\}) + \{w_k\} = \{u_k + v_k\} + \{w_k\} \quad \text{definition of addition in } S \\
= \{u_k + (v_k + w_k)\} \quad \text{definition of addition in } S \\
= \{u_k + (v_k + w_k)\} \quad \text{addition is associative in } \mathbb{R} \\
= \{u_k\} + \{v_k + w_k\} \quad \text{definition of addition in } S \\
= \{u_k\} + (\{v_k\} + \{w_k\}) \quad \text{definition of addition in } S
\]

4. Let \( \{0\} = \{\ldots, 0, 0, 0, 0, \ldots\} \) denote the doubly infinite sequence of zeros. Then \( \{u_k\} + \{0\} = \{u_k\} \) because:

\[
\{u_k\} + \{0\} = \{u_k + 0\} \quad \text{definition of addition in } S \\
= \{u_k\} \quad \text{property of real numbers}
\]
5. Let \(-\{u_k\} = (-1)\{u_k\}\). Then \(\{u_k\} + (-\{u_k\}) = \{0\}\) because:

\[
\begin{align*}
\{u_k\} + (-\{u_k\}) &= \{u_k\} + (-1)\{u_k\} \quad \text{definition of } -\{u_k\} \\
&= \{u_k\} + \{(1)u_k\} \quad \text{definition of scalar multiplication in } S \\
&= \{u_k\} + \{-u_k\} \quad \text{property of real numbers} \\
&= \{u_k + (-u_k)\} \quad \text{definition of addition in } S \\
&= \{0\} \quad \text{property of real numbers}
\end{align*}
\]

6. \(c\{u_k\}\) is in \(S\) since \(cu_k\) is a doubly infinite sequence.

7. \(c(\{u_k\} + \{v_k\}) = c\{u_k\} + c\{v_k\}\) because:

\[
\begin{align*}
c(\{u_k\} + \{v_k\}) &= c\{u_k + v_k\} \quad \text{definition of addition in } S \\
&= \{c(u_k + v_k)\} \quad \text{definition of scalar multiplication in } S \\
&= \{cu_k + cv_k\} \quad \text{distributive property of } \mathbb{R} \\
&= \{cu_k\} + \{cv_k\} \quad \text{definition of addition in } S \\
&= c\{u_k\} + c\{v_k\} \quad \text{definition of scalar multiplication in } S
\end{align*}
\]

8. \((c + d)\{u_k\} = c\{u_k\} + d\{u_k\}\) because:

\[
\begin{align*}
(c + d)\{u_k\} &= \{(c + d)u_k\} \quad \text{definition of scalar multiplication in } S \\
&= \{cu_k + du_k\} \quad \text{distributive property of } \mathbb{R} \\
&= \{cu_k\} + \{du_k\} \quad \text{definition of addition in } S \\
&= c\{u_k\} + d\{u_k\} \quad \text{definition of scalar multiplication in } S
\end{align*}
\]

9. \(c(d\{u_k\}) = (cd)\{u_k\}\) because:

\[
\begin{align*}
c(d\{u_k\}) &= c\{du_k\} \quad \text{definition of scalar multiplication in } S \\
&= \{c(du_k)\} \quad \text{definition of scalar multiplication in } S \\
&= \{(cd)u_k\} \quad \text{multiplication is associative in } \mathbb{R} \\
&= (cd)\{u_k\} \quad \text{definition of scalar multiplication in } S
\end{align*}
\]

10. \(1\{u_k\} = \{u_k\}\) because:

\[
\begin{align*}
1\{u_k\} &= \{1u_k\} \quad \text{definition of scalar multiplication in } S \\
&= \{u_k\} \quad \text{property of real numbers}
\end{align*}
\]

Since all ten vector space axioms hold, we can conclude that \(S\) is a vector space. \(\square\)
**Homework Exercise 1:**

\[ \mathbb{P}_n = \text{set of all polynomials of degree at most } n \text{ with real coefficients} \]

\[ = \{ a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n : a_1, a_2, \ldots, a_n \in \mathbb{R} \} \]

Given any two polynomials

\[ p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \text{ and } q(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n \]

and any scalar \( c \), define

\[ p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n \]

and

\[ c(p(t)) = (ca_0) + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n. \]

**Example:**

\[ p(t) = 1 + 2t + 3t^2, \quad q(t) = 2 + 4t, \quad c = 2 \]

\[ p(t) + q(t) = (1 + 2) + (2 + 4)t + (3 + 0)t^2 = 3 + 6t + 3t^2 \]

\[ 2(p(t)) = (2 \cdot 1) + (2 \cdot 2)t + (2 \cdot 3)t^2 = 2 + 4t + 6t^2 \]

**Claim:** The set \( \mathbb{P}_n \) of all polynomials of degree at most \( n \) with real coefficients is a vector space.

**Proof:** Let \( u(t) = u_0 + u_1 t + u_2 t^2 + \ldots + u_n t^n, v(t) = v_0 + v_1 t + v_2 t^2 + \ldots + v_n t^n, \) and \( w(t) = w_0 + w_1 t + w_2 t^2 + \ldots + w_n t^n \) be any three polynomials in \( \mathbb{P}_n \), and let \( c \) and \( d \) be any two scalars.

**Homework:** Fill in the details of the proof on a separate piece of paper. (For help see Example 4 on page 192.)

1. \( u(t) + v(t) \) is in \( \mathbb{P}_n \) since ???.
2. \( u(t) + v(t) = v(t) + u(t) \) because: ???
3. \( (u(t) + v(t)) + w(t) = u(t) + (v(t) + w(t)) \) because: ???
4. Let \( 0 \) denote the polynomial in \( \mathbb{P}_n \) with all coefficients equal to zero. Then \( u(t) + 0 = u(t) \) because: ???
5. Let \( -u(t) = (-1)u(t) \). Then \( u(t) + (-u(t)) = 0 \) because: ???
6. \( cu(t) \) is in \( \mathbb{P}_n \) since ???.
7. \( c(u(t) + v(t)) = cu(t) + cv(t) \) because: ???
8. \( (c + d)u(t) = cu(t) + du(t) \) because: ???
9. \( c(du(t)) = (cd)u(t) \) because: ???
10. \( 1u(t) = u(t) \) because: ???

Since all ten vector space axioms hold, we can conclude that \( \mathbb{P}_n \) is a vector space. ☐
**Homework Exercise 2:**

\[ \mathbb{F} = \text{set of all real-valued functions defined on a set } \mathbb{D} \]
\[ = \{ f : \mathbb{D} \to \mathbb{R} \} \]

Given any two functions \( f \) and \( g \), and any scalar \( c \), define \( f + g \) to be the function whose value at the point \( t \) in \( \mathbb{D} \) is \( f(t) + g(t) \), and define \( cf \) to be the function whose value at the point \( t \) in \( \mathbb{D} \) is \( c(f(t)) \).

Example: \( \mathbb{D} = (0, \infty) \), \( f(t) = t^2 \), \( g(t) = \ln(t) \), \( c = \frac{1}{2} \)

\[ (f + g) : (0, \infty) \to \mathbb{R} \text{ is defined by } (f + g)(t) = t^2 + \ln(t) \]
\[ \left( \frac{1}{2}f \right) : (0, \infty) \to \mathbb{R} \text{ is defined by } \left( \frac{1}{2}f \right)(t) = \frac{1}{2}t^2 \]

Note: to say \( f = g \) means \( f(t) = g(t) \) for all \( t \) in \( \mathbb{D} \).

**Claim:** The set \( \mathbb{F} \) of all real-valued functions defined on a set \( \mathbb{D} \) is a vector space.

**Proof:** Let \( f, g, \) and \( h \) be any three functions in \( \mathbb{F} \), and let \( c \) and \( d \) be any two scalars.

*Homework:* Fill in the details of the proof on a separate piece of paper. (For help see Example 5 on page 192.)

1. \( f + g \) is in \( \mathbb{F} \) since ???.
2. \( f + g = g + f \) because: ???
3. \( (f + g) + h = f + (g + h) \) because: ???
4. Let \( 0 \) denote the function whose value is zero for all \( t \) in \( \mathbb{D} \). Then \( f + 0 = f \) because: ???
5. Let \( -f = (-1)f \). Then \( f + (-f) = 0 \) because: ???
6. \( cf \) is in \( \mathbb{F} \) since ???.
7. \( c(f + g) = cf + cg \) because: ???
8. \( (c + d)f = cf + df \) because: ???
9. \( c(df) = (cd)f \) because: ???
10. \( 1f = f \) because: ???

Since all ten vector space axioms hold, we can conclude that \( \mathbb{F} \) is a vector space. \( \square \)
Homework Exercise 3:

\[ M_{2 \times 2} = \text{set of all } 2 \times 2 \text{ matrices with real entries} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\} \]

*Homework:* Show that the set \( M_{2 \times 2} \) of all \( 2 \times 2 \) matrices with real entries is a vector space.