Pattern Avoidance in Multiset Permutations: Bijective Proof

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1 Introduction

A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ of n letters *contains* the *pattern* $\tau = \tau_1\tau_2\ldots\tau_k$ of k letters if for some $i_1 < i_2 < \cdots < i_k$ we have $\sigma_{i_s} < \sigma_{i_t}$ whenever $\tau_s < \tau_t$. A permutation is said to *avoid* any pattern it does not contain. It is well-known that the number of permutations of n letters that avoid a pattern τ of 3 letters is independent of τ. Savage and Wilf [3] have shown the same result holds for permutations of a multiset. In this note we provide bijective proof of the Savage-Wilf result by generalizing the Simion-Schmidt [4] construction which established the theorem for (ordinary) permutations.

Savage and Wilf demonstrate that the number permutations of a multiset $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ avoiding a given pattern τ of 3 letters is a symmetric functions of the variables $k_1, k_2, \ldots, k_n$. Their proof extends theorems by Albert, Aldred, Atkinson, Handley, and Holton [1] and Atkinson, Linton, and Walker [2]. The bulk of their paper is dedicated to a bijective proof of the symmetry result for the pattern 123. The authors then note that their bijection can be extended to other patterns of 3 letters by means of the bijections described here in this note.

Simion and Schmidt construct lattices $L_n(\tau)$ whose covering relations are labeled with $1, 2, \ldots, n$ in such a way that along the different maximal chains, the labels form the τ-avoiding permutations exactly one time each. They then compare $L_n(123)$ with $L_n(132)$ to determine a bijection among 123- and 132-avoiding permutations. Their map, together with the reverse ($\sigma = \sigma_1\sigma_2\ldots\sigma_n \to \sigma^R = \sigma_n\sigma_{n-1}\ldots\sigma_1$) and complement ($\sigma = \sigma_1\sigma_2\ldots\sigma_n \to \sigma^C = (n-\sigma_1+1)(n-\sigma_2+1)\ldots(n-\sigma_n+1)$) maps, provides bijective proof that the number of permutations of n letters that avoid a pattern τ of 3 letters is independent of τ

In this note we provide a similar bijection for permutations of a multiset $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$. Below we construct a lattice $L_M(\tau)$, which we show is the poset of minimal cardinality whose covering relations are labeled with $1, 2, \ldots, n$ in such a way that along the different maximal chains, the labels form each of the τ-avoiding permutations of M exactly one time each. We compare the maximal chains of these lattices to determine bijections among both the 123- and 132-avoiding permutations of M, and the 123- and 213-avoiding

M-permutations. Since the complement of a given multiset permutation is generally not a permutation of the same multiset, we need the latter map to establish bijective proof that the number of permutations of a given multiset avoiding a pattern τ of 3 letters is independent of τ.

2 Lattices

Throughout this note M is the multiset $\{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$, and $A_M(\tau)$ denotes the set of τ-avoiding M-permutations. $L_M(\tau)$ is the lattice of minimal cardinality whose covering relations are labeled with $1, 2, \ldots, n$ in such a way that along the different maximal chains, the labels form each of the τ-avoiding M-permutations exactly once. The minimal cardinality requirement rules out, for example, the choice of $L_M(\tau)$ as the lattice consisting of a union of chains, one per permutation (with the proper labels), with the maximum and minimum elements identified. Following the construction of $L_M(\tau)$ described below, one might note that the lattice obtained is a subposet of a product of chains, a sort of generalized Boolean lattice, but we do not emphasize this point of view.

Construction of $L_n = L_M(123)$:

L_1 is the (k_1+1)-element chain with each covering labeled 1. To obtain L_m from L_{m-1}, $2 \le m \le n$, first construct $k_m + 1$ copies of the subposet T_{m-1} of L_{m-1} consisting of the chains in L_{m-1} whose labels read starting from $\hat{0}$ are nonincreasing, and are maximal with respect to this property. Note that as a graph, T_{m-1} is a spanning subtree of L_{m-1}. Each covering in T_{m-1} retains its label from L_{m-1}. Next linearly order the copies of T_{m-1}, and let each element in a particular copy of T_{m-1} cover its analog in the preceding copy and be covered by its analog in the succeeding copy. Label each of these $(k_1+1)(k_2+1)+\cdots+(k_{m-1}+1)k_m$ new coverings with m. Finally complete the top copy of T_{m-1} to L_{m-1}. See Figure 1.

We verify that labeled maximal chains in $L_M(123)$ correspond bijectively to 123-avoiding M-permutations as follows. The labels on the chain L_1 form the trivial 123-avoiding permutation of $\{1^{k_1}\}$. Assume $m \ge 2$ and that labeled maximal chains in L_{m-1} correspond bijectively to 123-avoiding permutations of $\{1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}}\}$. The paths in each tree T_{m-1} have nonincreasing labels. A "jump" in L_m from one copy of T_{m-1} to the next happens along edges labeled m, the largest label available at this stage of the construction. We imagine continuing along a path in the current copy of T_{m-1} as "picking up" where the path "left off" in the previous copy. Thus a maximal chain in L_m begins with a nonincreasing sequence of 1's, 2's, …, and $m-1$'s, into which k_m copies of m have been inserted. (These come from paths through the trees T_{m-1} connected by edges labeled m.) Following the final copy of m, the remaining 1's, 2's, …, and $m-1$'s form a 123-avoiding permutation. (The final m leads to the top copy of T_{m-1}, which is completed to L_{m-1}, and has maximal chains in bijection with 123-avoiding permutations by induction.) Before the

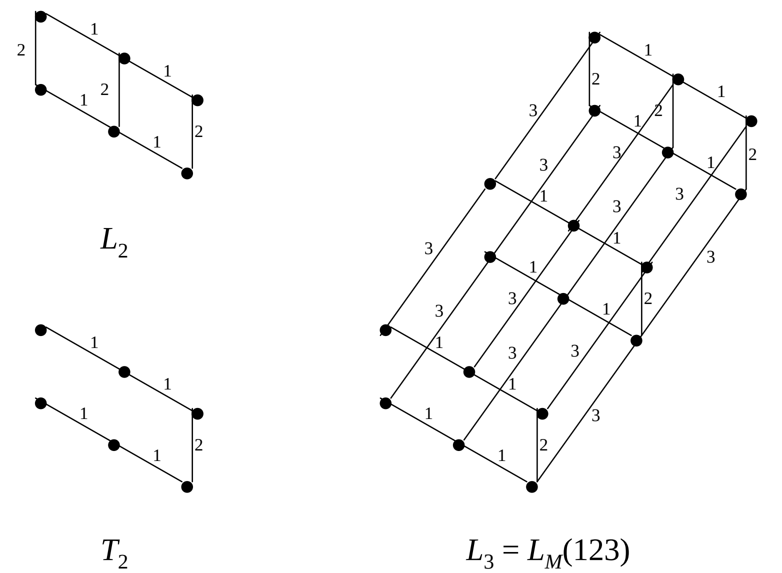

Figure 1: The construction of $L_M(123)$ for $M = \{1^2, 2^1, 3^2\}$.

last m there is no 123 pattern because the (largest) labels m are inserted into a nonincreasing sequence. After the last m there is no 123 pattern because this portion of the chain comes from labels in L_{m-1}. Finally we show there is no subsequence $l_1 < l_2 < l_3$ of labels on a maximal chain C with l_1 occurring before the last m, and l_3 appearing after the last m. Since we are assuming l_1 appears before the last m, we know it comes from from an edge e in one of the copies of T_{m-1}. Any edges in C which follow e in this (or any other) copy of T_{m-1} have labels at most l_1. Travel between copies of T_{m-1} occurs along edges labeled m, which cannot play the role of l_2. Thus an edge in C with label l_2 must occur in the top copy of T_{m-1} which has been completed to L_{m-1}. (An edge with label l_3 must also occur in this copy.) Since the construction of L_m begins with $k_m + 1$ copies of L_{m-1}, edge e from a particular copy of L_{m-1} (from which other edges have been removed to produce T_{m-1}) has an analog e' in the top copy of L_{m-1}, which also has label l_1. Since l_2 and l_3 come from edges in the top copy of L_{m-1}, we conclude that any occurrence of $l_1 < l_2 < l_3$ among the labels on C must also occur among the labels on a chain in L_{m-1}. But the labeled maximal chains in L_{m-1} correspond to 123-avoiding permutations of $\{1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}}\}$. Thus there can be no occurrence of $l_1 < l_2 < l_3$ among the labels on C, and we conclude that all labeled maximal chains in L_m must correspond to 123-avoiding permutations of $\{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$.

Next we show that a given 123-avoiding permutation σ of M arises from the sequence of labels on exactly one maximal chain in $L_M(123)$. Consider the portion of σ preceding the last copy of n. All parts less than n must occur in nonincreasing order to avoid a 123 pattern with n playing the role of the 3. For example the permutation $\sigma = 5433525115432121$ of $\{1^4, 2^3, 3^3, 4^2, 5^4\}$ has parts 4332211 in nonincreasing order preceding the last copy of $n = 5$. This portion of σ comes from a path through the first three copies of T_4 linked by edges labeled 5 in $L_5 = L_M(123)$. The remaining portion 432121 of σ comes from a chain in the top copy of T_4 which is completed to L_4. By induction L_4 contains all 123-avoiding permutations of $\{1^4, 2^3, 3^3, 4^2\}$. In particular it contains 4332211432121 (σ with the 5's removed). The last 5 in σ comes from an edge in L_5 which leads from the third copy of T_4 to L_4. It leads to the vertex found between the consecutive edges labeled 1 and 4 in the chain labeled 4332211432121 in L_4. Thus σ arises from a maximal chain in $L_M(123)$. To show it arises from a unique maximal chain, we observe that if it arose from two chains, then both of these chains with then n's removed would occur in L_{n-1}. But L_{n-1} has by induction has maximal chains in bijection with the 123-avoiding permutations of $\{1^{k_1}, 2^{k_2}, \ldots, (n-1)^{k_{n-1}}\}$. Thus σ must arise from the labels on a unique chain in $L_M(123)$ as well.

That $L_M(123)$ is a lattice also follows by induction. Since L_1 is a chain, it is trivially a lattice. Let $m \geq 2$ and suppose L_{m-1} is a lattice. Since L_m has a maximum element, it is enough to determine the meet $x \wedge y$ of any two elements x and y in the poset. Now x is an element of one of the $k_m + 1$ copies of T_{m-1} used to construct L_m, and so is y. Suppose x is in copy c_x and y is in copy c_y, with $c_x \leq c_y$. If x and y are both in the top copy of T_{m-1}, which has been completed to L_{m-1}, then $x \wedge y$ in L_m is $x \wedge y$ in L_{m-1} since the L_{m-1} is a lattice. If x is not in the top copy of T_{m-1}, then we will see that $x \wedge y$ is an element in copy c_x of T_{m-1}. Let y' denote the analog of y in the c_x copy of T_{m-1}. If $c_x = c_y$, then $y = y'$, otherwise $y' < y$ since the two elements are linked by a sequence of edges labeled m. Since the Hasse diagram of T_{m-1} is a tree oriented away from one vertex, it has a well-defined meet operation, and $x \wedge y$ in L_m is $x \wedge y'$ in copy c_x of T_{m-1}. More specifically, we know T_{m-1} is composed of $k_{m-1}+1$ linearly ordered copies of T_{m-2}: the $\hat{0}$ element of a given copy of T_{m-2} is linked to the $\hat{0}$ element of the following copy by an edge labeled $m-1$. Thus if x and y' appear in distinct copies c_x and $c_{y'}$ of the tree T_{m-2}, then $x \wedge y = x \wedge y'$ is the $\hat{0}$ element of copy $\min\{c_x, c_{y'}\}$ of T_{m-2}. If x and y' appear in the same copy of T_{m-2}, on the other hand, then we must view T_{m-2} as constructed from $k_{m-2}+1$ linearly ordered copies of T_{m-3} linked by their $\hat{0}$ elements, and so forth. If we reach a minimum i, $1 \leq i \leq m-2$, for which x and y' appear in distinct copies of T_{m-1-i}, say copies c_x and $c_{y'}$, respectively, then $x \wedge y = x \wedge y'$ is the $\hat{0}$ element of $\min\{c_x, c_{y'}\}$. If we don't reach such an i, then x and y' appear in the same copy of T_1, and $x \wedge y' = y'$.

To justify the claim that $L_M(123)$ is the lattice of minimal cardinality whose covering relations are labeled with $1, 2, \ldots, n$ in such a way that along the different maximal chains, the labels form each of the 123-avoiding M-permutations exactly once, we observe that the proof given by Simion and Schmidt for the case $k_1 = k_2 = \cdots = k_n = 1$ can be generalized to accommodate $k_i \geq 1$ for $1 \leq i \leq n$. Simion and Schmidt show that any lattice with the desired properties for (ordinary) permutations has at least 2^n vertices (elements) and at least $2^{n+1} - n - 2$ edges (covering relations). They then remark that their lattices have the minimum number of vertices and edges. Their proof can be generalized to show that any lattice with the desired properties for multiset permutations has at least $(k_1 + 1)(k_2 + 1) \dots (k_n + 1)$ vertices and at least $2(k_1 + 1)(k_2 + 1) \dots (k_n + 1) - (k_1 + k_2 + \cdots + k_n) - 2$ edges. $L_M(123)$ has the minimum number of vertices and edges.

Construction of $L_n = L_M(132)$:

L_1 is again the $(k_1 + 1)$-element chain with each covering labeled 1. We obtain L_m, $2 \leq m \leq n$, from $k_m + 1$ copies of L_{m-1}, each with its own labeling, together with $(1 + k_1 + k_2 + \cdots + k_{m-1})k_m$ additional coverings labeled m. First linearly order the copies of L_{m-1}. Then locate on each copy of L_{m-1} the maximal chain consisting of coverings with nonincreasing labels when read in order starting from $\hat{0}$. Next let each element of such a maximal chain cover its analog in the preceding copy of L_{m-1} and be covered by its analog in the succeeding copy. Label each new covering m. See Figure 2.

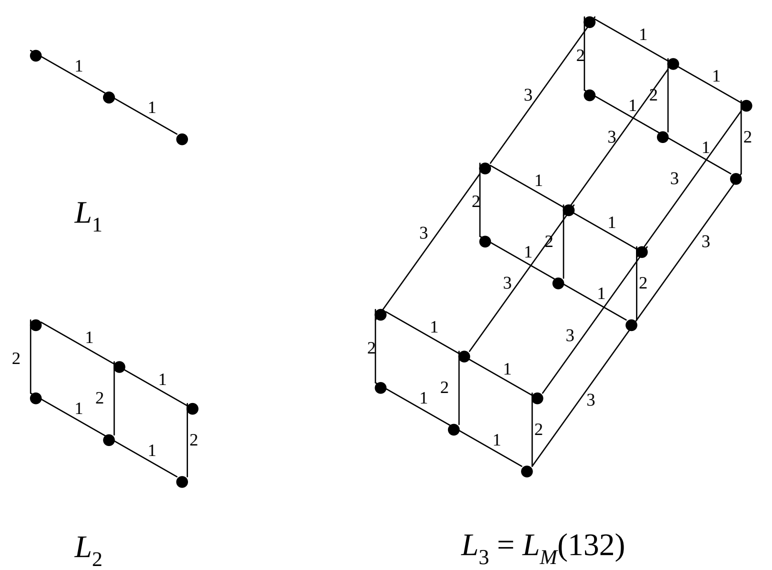

Figure 2: The construction of $L_M(132)$ for $M = \{1^2, 2^1, 3^2\}$.

Construction of $L_n = L_M(213)$:

The construction of $L_n(213)$ is exactly the same as the construction of $L_n(123)$ with the exception that $T_m(213)$ consists of the chains in $L_m(213)$ whose labels are nondecreasing when read starting from $\hat{0}$. See Figure 3.

The inductive construction of $L_M(132)$ guarantees that along each maxi-

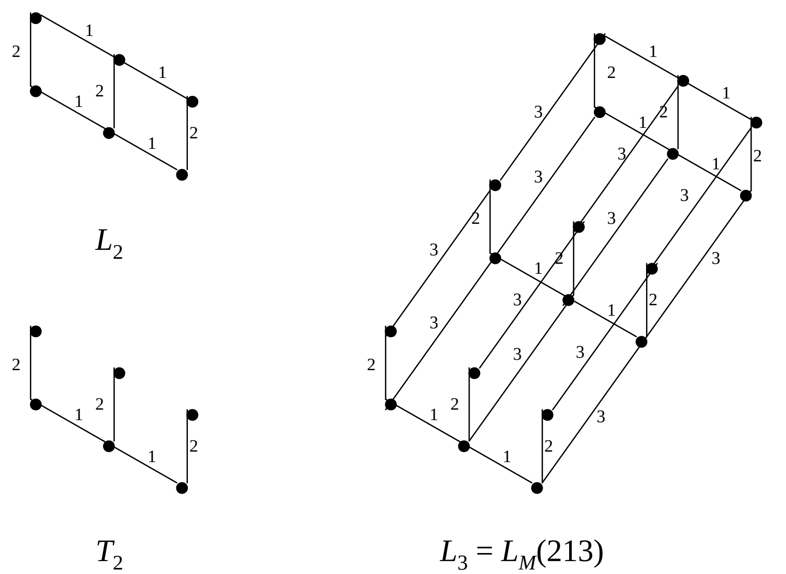

Figure 3: The construction of $L_M(213)$ for $M = \{1^2, 2^1, 3^2\}$.

mal chain, the labels m have been inserted into a 132-avoiding permutation of $\{1^{k_1}, 2^{k_2}, \ldots, (m-1)^{k_{m-1}}\}$. Every label preceding a given m is at least as large as every label succeeding it. Thus the labels on maximal chains in $L_M(132)$ avoid the pattern 132. Analogous statements hold for $L_M(213)$. Since the construction of $L_M(213)$ is so similar to that of $L_M(123)$, the proof that $L_M(213)$ is a lattice is almost identical to that for $L_M(123)$. The construction for $L_M(132)$, on the other hand, is different from that of the other two lattices. We therefore include justification that it is a lattice below. Proofs that $L_M(132)$ and $L_M(213)$ are the lattices of minimal cardinality with labeled maximal chains in bijection with the 132- and 213-avoiding M-permutations, respectively, also follows from their constructions, and are similar to that given for $L_M(123)$ above.

That $L_n = L_M(132)$ is a lattice follows from its inductive construction. Again L_1 is a trivial lattice, and we assume L_{m-1} is a lattice for $m \geq 2$. Since L_m has a maximum element, we determine the meet $x \wedge y$ of any two elements x and y in L_m. Now L_m is constructed from k_m+1 linearly ordered copies of L_{m-1}. In each copy the maximal chain consisting of coverings with nonincreasing labels when read starting from $\hat{0}$ is highlighted. Each element of this maximal chain is linked to its analog in the following copy of L_{m-1}. If x and y both appear in the same copy of L_{m-1}, we have $x \wedge y$ in L_m equal to $x \wedge y$ in L_{m-1} since L_{m-1} is a lattice. Suppose x and y appear in distinct copies c_x and c_y, $c_x < c_y$, respectively, of L_{m-1}. We will see that $x \wedge y$ is an element in copy c_x of L_{m-1}. Now y is comparable to one or more elements of the highlighted maximal chain in copy c_x of L_{m-1}. Let y' be the greatest such element. (Note y' is obtained by traveling from y in copy c_y of L_{m-1} along edges with labels which are as large as possible to elements with which it is comparable. When an element along the highlighted maximal chain in copy c_y is reached, y' is the analog of this element in copy c_x of L_{m-1}. The element y' is reached by following a sequence of edges labeled m connecting the various copies of L_{m-1}.) Since L_{m-1} is a lattice, we have an element $x \wedge y'$ in L_{m-1}. This element is $x \wedge y$ in L_m.

3 Bijections for 123- and 132-avoiding multiset permutations

Examination of the lattices constructed in the previous section yields bijections between the 123- and 132-avoiding multiset permutations, $A_M(123)$ and $A_M(132)$, and the 123- and 213-avoiding multiset permutations, $A_M(123)$ and $A_M(213)$. In this section we construct inverse maps between $A_M(123)$ and $A_M(132)$. To do this we make only minor modifications to those given by Simion and Schmidt for the case $k_1 = k_2 = \cdots = k_n = 1$.

Let $\alpha = \alpha_1\alpha_2\ldots\alpha_N$, $N = \sum k_i$, be a 123-avoiding M-permutation. We visualize the map which takes α to a 132-avoiding M-permutation $\beta = \beta_1\beta_2\ldots\beta_N$ by imagining α as a labeled maximal chain in $L_M(123)$. Beginning with the $\hat{0}$ element in $L_M(132)$, we attempt to follow α in this lattice. When α leads to an edge which doesn't exist in $L_M(132)$, we instead follow the edge with the smallest label which is larger than the label on our current edge. For the inverse map, we attempt to follow β in $L_M(123)$. When β leads to a nonexistent edge, we choose to follow the edge with the largest label. For example consider $\alpha = 3214312$ in $L_M(123)$, $M = \{1^2, 2^2, 3^3, 4\}$. Since M contains one copy of 4, we know $L_4(123) = L_M(123)$ is constructed from two copies of $L_3(123)$, the first of which has had edges removed to form $T_3(123)$. Each vertex in $T_3(123)$ is covered by the corresponding vertex in $L_3(123)$ by an edge labeled 4. Because α begins in $T_3(123)$, the labels are nonincreasing until the 4 is reached. The nonincreasing portion of α can be followed in $L_M(132)$. The 4, however, cannot be followed in $L_M(132)$ at this point. $L_M(132)$ is composed of two (complete) copies of of $L_3(132)$, with only certain corresponding pairs of vertices (those on the "right" of $L_3(132)$) joined by an edge labeled 4. We must "get right" in $L_3(132)$ before we can follow an edge labeled 4. We do this by following edges with nondecreasing labels that appear in α, the first of which is strictly larger than the label on the current edge (which is 1 in this example), until we can follow the 4. See Figures 4 and 5.

Proposition 1 *Given a multiset $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ with $k_i \geq 1$, there exist inverse maps $\mathcal{A} : A_M(123) \longrightarrow A_M(132)$ and $\mathcal{B} : A_M(132) \longrightarrow A_M(123)$.*

Proof. Let $\alpha = \alpha_1\alpha_2\ldots\alpha_N$, $N = \sum k_i$, be a 123-avoiding M-permutation. To construct the 132-avoiding M-permutation $\beta = \beta_1\beta_2\ldots\beta_N = \mathcal{A}(\alpha)$ we do the following:

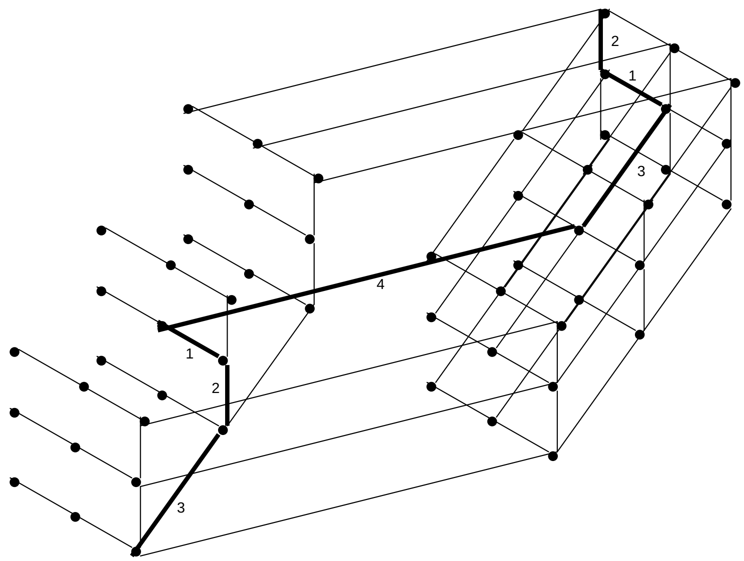

Figure 4: The chain $\alpha = 3214312$ in $L_M(123)$, $M = \{1^2, 2^2, 3^3, 4\}$. (Several edges labeled 4 have been omitted for purposes of clarity.)

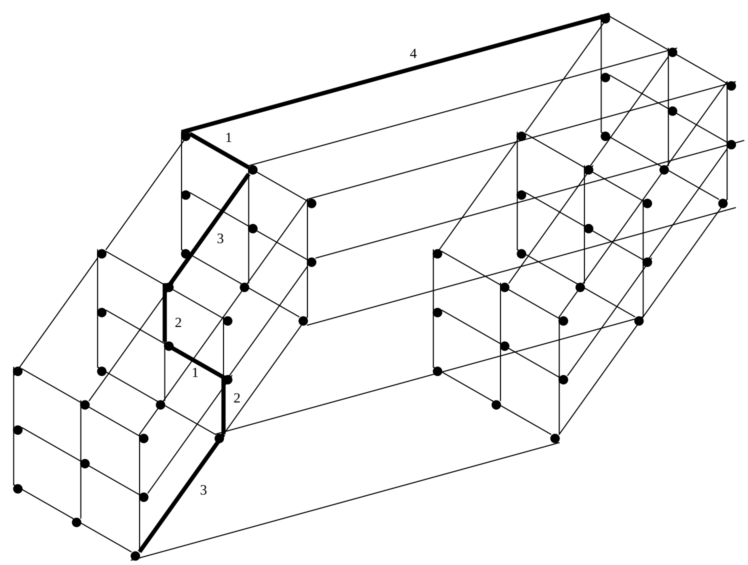

Figure 5: The chain $\beta = 3212314$ in $L_M(132)$, $M = \{1^2, 2^2, 3^3, 4\}$.

1. Set $i = 1$, $\beta_1 = \alpha_1$, and $x = \alpha_1$.
2. Increment i. If $i > N$, then end. Otherwise go to step 3.
3. If $\alpha_i \leq x$, then set $\beta_i = \alpha_i$, assign $x = \alpha_i$, and go to step 2. Otherwise set $\beta_i = \min\{k : k > x, k \in M - \{\beta_1, \beta_2, \ldots, \beta_{i-1}\}\}$ and go to step 2.

Now let $\beta = \beta_1\beta_2 \ldots \beta_N$ be a 132-avoiding M-permutation. To construct the 123-avoiding M-permutation $\alpha = \alpha_1\alpha_2 \ldots \alpha_N = \mathcal{B}(\beta)$ we do the following:

1. Set $i = 1$, $\alpha_1 = \beta_1$, and $x = \beta_1$.
2. Increment i. If $i > N$, then end. Otherwise go to step 3.
3. If $\beta_i \leq x$, then set $\alpha_i = \beta_i$, assign $x = \beta_i$, and go to step 2. Otherwise set $\alpha_i = \max\{k : k \in M - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}\}$ and go to step 2.

To verify $\beta = \mathcal{A}(\alpha)$ is in $A_M(132)$ and $\mathcal{B}(\beta) = \alpha$, we first observe the existence of a maximal subsequence $\alpha_1 = \alpha_{i_1} \geq \alpha_{i_2} \geq \cdots \geq \alpha_{i_l}$ of letters in α which is fixed by $\mathcal{A}$, i.e., $\beta_{i_k} = \alpha_{i_k}$ for $1 \leq k \leq l$. This subsequence is also fixed by $\mathcal{B}$. (Note that $\alpha_{i_k} = \beta_{i_k}$, $1 \leq k \leq l$, is the sequence of values assigned to x by both algorithms.) Next we let α_j be any letter of α which is not fixed by $\mathcal{A}$. At step j of the algorithm for $\mathcal{A}$, the letter α_j is compared with x, the smallest letter to its left. Since α_j is not fixed by $\mathcal{A}$, we know $x < \alpha_j$. We also know α_j must be the largest letter greater than x in $M - \{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}\}$ (otherwise x plays the role of the 1 and α_j plays the role of the 2 in an occurrence of the pattern 123). Then we see there is only one choice for β_j that guarantees β avoids 132; β_j must be the smallest letter in $M - \{\beta_1, \beta_2, \ldots, \beta_{j-1}\}$ (otherwise x plays the role of the 1 and β_j plays the role of the 3 in an occurrence of the pattern 132). Thus β is in $A_M(132)$ and $\mathcal{B}(\beta) = \alpha$, as desired. ■

4 Bijections for 123- and 213-avoiding multiset permutations

Next we establish injections $\mathcal{A} : A_M(123) \to A_M(213)$ and $\mathcal{B} : A_M(213) \to A_M(123)$. The intuition behind the algorithms describing these maps comes from comparing the lattices $L_M(123)$ and $L_M(213)$. We imagine a maximal chain α in $L_M(123)$ "jumping" between successive copies of $T_{n-1}(123)$ along covering relations labeled n. The last n leads to the final copy of $T_{n-1}(123)$, which is completed to $L_{n-1}(123)$. Consider the portion of α preceding the last n. Let v denote the vertex of $T_{n-1}(123)$ leading to $L_{n-1}(123)$ via the last edge labeled n. When the $k_n - 1$ remaining values n are removed from the portion of α under consideration, the resulting sequence describes the unique path from $\hat{0}$ to v in the tree $T_{n-1}(123)$. From the construction of $T_{n-1}(123)$ and $T_{n-1}(213)$, we see that both posets can be regarded as having the same ground set, but differing covering relations. Thus we can refer to the analog v' in $T_{n-1}(213)$ of

v in $T_{n-1}(123)$. There is a unique path from $\hat{0}$ to v' in $T_{n-1}(213)$ as well. This observation together with the inductive construction of the lattices yields the algorithm.

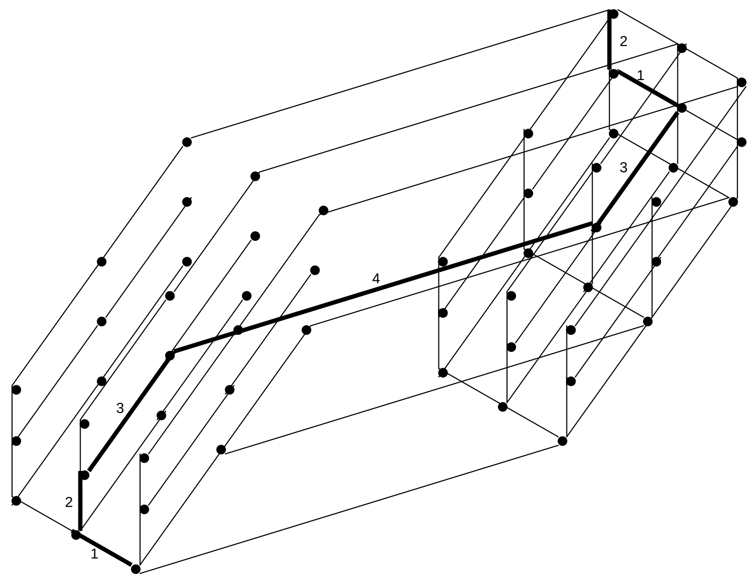

Figure 6: The chain $\beta = 1234312$ in $L_M(213)$, $M = \{1^2, 2^2, 3^3, 4\}$. (Several edges labeled 4 have been omitted for purposes of clarity.)

Proposition 2 *Given a multiset* $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ *with* $k_i \geq 1$, *there exist injections* $\mathcal{A} : A_M(123) \to A_M(213)$ *and* $\mathcal{B} : A_M(213) \to A_M(123)$.

Proof. First we describe the maps $\mathcal{A}$ and $\mathcal{B}$, then we show they are injective.

Let $\alpha = \alpha_1\alpha_2\ldots\alpha_N$, $N = \sum k_i$, be a 123-avoiding M-permutation. To construct the 213-avoiding M-permutation $\beta = \beta_1\beta_2\ldots\beta_N = \mathcal{A}(\alpha)$ we do the following:

1. Set $i = 3$ and *temp* $= \alpha$.

2. If $i > n$, then set $\beta =$ *temp* and end.

3. Fix all parts greater than or equal to i in *temp*.

4. Locate the rightmost part i in *temp*, and fix all parts to the right of this i.

5. Place all unfixed parts in *temp* in nondecreasing order.

6. Increment i and go to step 2.

To obtain the map $\mathcal{B}$ simply replace the word "nondecreasing" in step 5 by "nonincreasing".

To see that β is indeed a 213-avoiding M-permutation, we choose a particular copy of any letter l, $l < n$, to play the role of the 2. Let m denote the rightmost letter in α which is strictly greater than l. When the algorithm for $\mathcal{A}$ reaches the step where $i = m$, all letters less than m (including l) to the left of m are placed in nondecreasing order, while all letters to the right of m are fixed. At this and all subsequent steps of the algorithm, l is not greater than any letter which appears both to its right and to the left of m. Thus any letter which can play the role of the 1 must appear to the right of m. But since m is the rightmost letter greater than l, there then remains no letter to play the role of the 3. Thus β is a 213-avoiding M-permutation. A similar argument shows $\mathcal{B}(\gamma)$ is a 123-avoiding M-permutation for any γ in $A(213)$.

Next we show $\mathcal{A}$ and $\mathcal{B}$ are injective. Suppose $\mathcal{A}(\alpha_A) = \mathcal{A}(\alpha_B)$. Then α_A and α_B must have all copies of n in the same positions. Let α'_A and α'_B denote the portions of α_A and α_B, respectively, to the right of the rightmost copy of n. If $\alpha'_A = \alpha'_B$, then $\alpha_A = \alpha_B$ since all parts of α_A and α_B which are less than n and to the left of the rightmost copy of n appear in nonincreasing order. Since $\mathcal{A}(\alpha_A) = \mathcal{A}(\alpha_B)$ we know α'_A and α'_B must have all copies of $n-1$ (if any) in the same positions. Let α''_A and α''_B denote the portions of α'_A and α'_B, respectively, to the right of the rightmost copy of $n-1$. If $\alpha''_A = \alpha''_B$, then $\alpha'_A = \alpha'_B$ since all parts of α'_A and α'_B which are less than $n-1$ and to the left of the rightmost copy of $n-1$ appear in nonincreasing order. Since this process cannot continue indefinitely, we conclude $\alpha_A = \alpha_B$. A similar argument shows $\mathcal{B}$ is injective. ■

As an example consider $\alpha = 3214312$ pictured in Figure 4. The image of α under the map $\mathcal{A}$ described above is $\beta = 1234312$ shown in Figure 6. Note that $\mathcal{A}$ and $\mathcal{B}$ are not inverse maps. This can be seen by letting $\alpha = 32143124$, for example. Then $\beta = \mathcal{A}(\alpha) = 11242334$, but $\mathcal{B}(\beta) = 33242114$ is not α.

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