Rollette

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Abstract

For his senior capstone project at St. Joseph's University in Philadelphia, Joe Witiw created a new casino game called *Rollette*. To begin this game, a player places a bet on one of the numbers in the set $S = \{20, 30, 40, 50\}$. The player then rolls two dice repeatedly, keeping track of the cumulative sum with each roll. If he reaches his bet exactly, the player wins. If he overshoots his bet, he loses. To add an interesting twist to this otherwise straightforward game, the player also loses if he reaches exactly half his bet. How much money should a casino pay to a player who wins after placing a \$1 bet on a particular element of S? The payout should be high enough to entice players, but low enough to guarantee the casino a profit in the long run.

1 Introduction.

When a casino chooses the payout for a win on a particular bet of a given game of chance, it typically selects an amount that guarantees the house an average gain of \$0.05 (per \$1 bet) per game in the long run. According to the *Law of Large Numbers*, the proportion of times a player wins a given game with a particular bet approaches the theoretical probability of doing so as the number of plays increases. In order to select an appropriate payout, a casino must determine this theoretical probability.

Here we determine the probabilities and payouts for a new casino game invented by a college senior as part of his capstone project. Joe Witiw of St. Joseph's University proposed a game he callled *Rollette*, in which a player places a bet on one of the numbers in the set $S = \{20, 30, 40, 50\}$. The player then rolls two dice repeatedly, keeping track of the cumulative sum with each roll. If he reaches his bet exactly, the player wins. If he overshoots his bet, he loses. To add interest to the game, the player also loses if he reaches exactly half his bet.

2 Probabilities.

Let $P(n)$ denote the probability of obtaining a cumulative sum of exactly n in repeated rolls of two fair dice. For the moment, we set aside the twist in

Rollette which causes a player to lose upon reaching half his bet, and focus the probability of rolling a sum of n.

We begin by computing $P(n)$ directly for small values of n. The increasing complexity of the computations will soon motivate a different approach.

We know $P(1) = 0$ since a sum of 1 is impossible to achieve with a roll of two dice. A sum of 2 or 3 can be obtained only by a single roll of two dice, whereas a sum of 4 or more can be obtained with more than one roll, and a sum greater than 12 requires at least two rolls. We have $P(2) = 1/36 \approx 0.028$, $P(3) = 2(1/36) \approx 0.056$, and $P(4) = 3(1/36) + 1(1/36)^2 \approx 0.084$. (We obtain a sum of 4 with either $(1, 3)$, $(2, 2)$, or $(3, 1)$ as the outcome of a single roll, or $(1, 1)$ and $(1, 1)$ as the outcome of two rolls. Similarly $P(5) = 4(1/36) + 4(1/36)^2 \approx 0.114$. (We obtain a sum of 5 with either $(1, 4)$, $(2, 3)$, $(3, 2)$, or $(4, 1)$ as the outcome of one roll, or $(1, 1)$ and $(1, 2)$, $(1, 1)$ and $(2, 1)$, $(1, 2)$ and $(1, 1)$, or $(2, 1)$ and $(1, 1)$ as the outcome of two rolls.) Can you obtain $P(6) = 5(1/36) + 11(1/36)^2 + 3(1/36)^3 \approx 0.147$ and $P(7) = 6(1/36) + 20(1/36)^2 + 6(1/36)^3 \approx 0.182$?

Any reader persistent enough to compute $P(8) \approx 0.166$ directly will welcome a new strategy for arriving at $P(n)$. When direct computation becomes unwieldy, a generating function approach may offer relief.

To determine the probability $P(n)$ of obtaining a cumulative sum of exactly n in repeated rolls of two dice, consider the generating function

$$F(z) = \sum_{n \geq 0} P(n) z^n$$

where we define $P(0) = 1$. $F(z)$ is the formal power series in which the coefficient of z^n gives us the probability we seek. Below we find a simple expression for $F(z)$ as a quotient of polynomials in the variable z, then use polynomial long division to compute the coefficient of z^n, which is the desired $P(n)$. In a later section we show $F(z)$ is also a function of the complex variable z for $|z| < 1$. This additional information allows us to use Taylor's theorem to obtain $P(n)$.

Let $G(z)$ denote the generating function

$$\begin{aligned} G(z) &= \frac{1}{6}z + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6 \\ &= \frac{z^7 - z}{6(z-1)} \end{aligned}$$

in which the coefficient of z^n gives the probability of obtaining n with a single roll of one die.

Since the coefficient of z^n in $G(z)$ is the probability of obtaining n with a single roll of one die, and since successive rolls of a die are independent, we know the coefficient of z^n in

$$\begin{aligned} H(z) &= G(z)^2 \\ &= \left(\frac{z^7 - z}{6(z-1)} \right)^2 \end{aligned}$$

is the probability of obtaining a sum of n with a single roll of two dice (or two rolls of one die).

Because the coefficient of z^n in $H(z)$ is the probability of obtaining a sum of n with a single roll of two dice, we know the coefficient of z^n in $H(z)^k$ is the probability of obtaining a sum of n with exactly k rolls of two dice.

Since $P(n)$ is the probability of obtaining a sum of n with any number of rolls (one or two or three...) of two dice, we have $F(z) = 1 + H(z) + H(z)^2 + H(z)^3 + \ldots$, *i.e.*,

$$\begin{aligned} F(z) &= \sum_{k\geq 0} H(z)^k \\ &= \frac{1}{1-H(z)} \\ &= \frac{1}{1-(\frac{z^7-z}{6(z-1)})^2} \\ &= -\frac{36z^2-72z+36}{z^{14}-2z^8-35z^2+72z-36}. \end{aligned}$$

We can now use the formula for $F(z)$ to determine $P(n)$ using polynomial long division. In so doing we obtain the sequence

$$\{P(n)\}_{n\geq 2} \approx 0.028, 0.056, 0.084, 0.114, 0.147, 0.182, 0.166, 0.156, 0.148\ldots.$$

Because $F(z)$ also turns out to be a function of z for $|z| < 1$ (shown below), we may also use the Maclaurin series for $F(z)$ to obtain these numbers.

3 Payouts.

Now that we have the probabilities we need, how can we use them to select appropriate payouts for *Rollette*? We begin by considering an example featuring a simpler game with a similar name: *Roulette* pays \$35 for a winning \$1 bet on a single number. Since an American wheel has 38 slots (1 − 36, 0, and 00), the theoretical probability of winning on a single number bet is 1/38. After a large number of such bets, the house expects to make an average of $(37/38) \times \$1 - (1/38) \times \$35 \approx \$0.05$ per game. For most plays, the house collects \$1, but sometimes it returns the \$1 and pays \$35. The *Law of Large Numbers* guarantees that the proportion of times a casino must pay \$35 is close to 1/38 since a very large number of *Roulette* plays will occur over time.

We now return to *Rollette*. How should we assign payouts to the various bets in our game so that the house expects to win at least \$0.05 per game in the long run? Recall the set $S = \{20, 30, 40, 50\}$ of possible bets. We can use the generating function $F(z)$ to determine the probability $P(n)$ of reaching a sum of n for any bet n in S, but what about the twist that causes a player to lose if he reaches exactly half his bet? Should the twist raise or lower the payout?

Let $p(n, m)$ denote the probability of rolling a cumulative sum of exactly n in repeated rolls of two fair dice *without* first reaching a sum of exactly m. To determine this number, we first compute the probability of rolling a sum of m, then continuing to also roll a sum of n. We know the probability of rolling a sum of m with a sequence of rolls of two dice is $P(m)$. To continue on to also roll a sum of n, we need an additional sequence of rolls with a sum of $n - m$. The probability of this second sequence of rolls is $P(n - m)$. The probability that the two sequences occur together in the right order is $P(m)P(n - m)$. This means $P(n) = p(n, m) + P(m)P(n - m)$ (to roll a sum of n, either we reach a sum of m on the way or we don't). From this observation we obtain $p(n, m) = P(n) - P(m)P(n - m)$.

For *Rollette* we're interested in $p(n, n/2)$ for n in S. As an example let us determine an appropriate payout for a \$1 bet on $n = 20$. *Mathematica* tells us $P(20) \approx 0.140764$ and $P(10) \approx 0.147784$, so $p(20, 10) \approx 0.118924$. Let x be the payout for a bet on $n = 20$. We need $(1 - 0.118924) \times \$1 - 0.118924 \times \$x \geq \$0.05$. Solving this inequality yields $x \leq 7.40874$. This means any payout less than \$7.41 for a winning bet on $n = 20$ guarantees the house at least \$0.05 per play in the long run. (For the sake of esthetics, a casino might choose to make the payout \$7 exactly.) Since $P(50)$ doesn't differ greatly from $P(20)$, we would expect an appropriate payout for a bet on $n = 50$ to be similar to that computed above. In this case the amount is at most \$6.77 (see Figure 1). Note that without the twist, the payout for a \$1 bet on $n = 20$ should satisfy $(1 - P(20)) \times \$1 - P(20) \times \$x \geq \$0.05$. In this case the payout should be at most \$5.75.

n	Payout
20	\$7.41
30	\$6.83
40	\$6.72
50	\$6.77

Figure 1: Maximum payout for a winning bet on n that guarantees an average of at least \$0.05 per game profit for the house in the long run.

4 Convergence of $P(n)$.

Since the values $P(n)$ for n in S are so close together, it's natural to suspect that the sequence $\{P(n)\}_{n \geq 2}$ converges. Does it? We did not need the answer to determine payouts for *Rolette*, but the question is interesting, so we'll answer it.

Recall from calculus that $1 + x + x^2 + x^3 + \ldots$ converges to $\frac{1}{1-x}$ whenever

$|x| < 1$. Here we're interested in $x = H(z)$. By the triangle inequality we know

$$\begin{aligned}
|H(z)| &= |\tfrac{z^7-z}{6(z-1)})^2| \\
&= |\tfrac{1}{6}(z+z^2+z^3+z^4+z^5+z^6)^2| \\
&= |\tfrac{1}{36}z^2 + \tfrac{2}{36}z^3 + \tfrac{3}{36}z^4 + \ldots + \tfrac{1}{36}z^{12}| \\
&\leq \tfrac{1}{36}|z|^2 + \tfrac{2}{36}|z|^3 + \tfrac{3}{36}|z|^4 + \ldots + \tfrac{1}{36}|z|^{12}.
\end{aligned}$$

This means $|H(z)| < 1$ whenever $|z| < 1$. We have shown that

$$\begin{aligned}
F(z) &= \frac{1}{1 - \left(\frac{z^7-z}{6(z-1)}\right)^2} \\
&= -\frac{36z^2 - 72z + 36}{z^{14} - 2z^8 - 35z^2 + 72z - 36}
\end{aligned}$$

is a function of the complex variable z when $|z| < 1$.

Below we use the fact that $F(z)$ has a simple pole at $z = 1$ with residue $-\frac{1}{7}$ to show

$$\lim_{n\to\infty} P(n) = \frac{1}{7} \approx 0.143.$$

Let

$$D(z) = 1 - H(z)$$

denote the denominator of $F(z)$. Since $D(1) = 0$, we know $F(z)$ has a singular point at $z = 1$, and we can find its Laurent series about that singular point. To do so, we begin with the series representation for $D(z)$. Because $D(z)$ is analytic everywhere, it has a Taylor series about $z = 1$.

$$D(z) = D(1) + D'(1)(z-1) + \frac{D''(2)}{2}(z-1)^2 + \ldots$$

Since $D(1) = 0$, we know $(z-1)$ divides evenly into $D(z)$. Since $D'(1) = -H'(1) = -7$, we know $(z-1)^2$ does not divide evenly into $D(z)$. Thus when we multiply $F(z)$ by $(z-1)$, we obtain a function which is analytic everywhere (the factor of $(z-1)$ in the denominator cancels with the multiplier), and therefore has a Taylor series about $z = 1$.

$$(z-1)F(z) = \sum_{n\geq 0} a_n (z-1)^n$$

This yields a Laurent series for $F(z)$.

$$F(z) = \frac{a_0}{z-1} + a_1 + a_2(z-1) + a_3(z-1)^2 + \ldots$$

We see that $F(z)$ has a simple pole at $z = 1$, and calculate its residue there. We have $(z-1)F(z) = (z-1)/D(z)$, and we know $(z-1)$ divides $D(z)$. Factoring $(z-1)$ out of the denominator, canceling it with that in the numerator, and substituting $z = 1$ yields $a_0 = -\frac{1}{7}$.

$$F(z) = \frac{-\frac{1}{7}}{z-1} + \sum_{n\geq 0} a_{n+1}(z-1)^n$$

Let $J(z) = \dfrac{-\frac{1}{7}}{z-1}$ so that $F(z) = J(z) + \sum_{n\geq 0} a_{n+1}(z-1)^n$, and

$$J(z) = \frac{-\frac{1}{7}}{z-1} = \frac{\frac{1}{7}}{1-z} = \frac{1}{7}(1+z+z^2+\ldots) = \frac{1}{7} + \frac{1}{7}z + \frac{1}{7}z^2 + \ldots.$$

Recalling the definition $F(z) = \sum_{n\geq 0} P(n)z^n$, we have

$$\sum_{n\geq 0}\left[P(n) - \frac{1}{7}\right] z^n = F(z) - J(z) = \sum_{n\geq 0} a_{n+1}(z-1)^n,$$

which converges for some z with $|z| = 1$. Thus $\lim_{n\to\infty} |P(n) - \frac{1}{7}| = 0$, *i.e.*,

$$\lim_{n\to\infty} P(n) = \frac{1}{7}.$$

5 Formula.

Can we find an exact formula for $P(n)$? When repeated rolls of two dice determine this probability, as in the game described above, an exact formula is hard to come by. When we use a single die, however, we can find an exact formula. We can then interpret an even number of rolls of a single die as half the number of rolls of two dice.

Now we let $P(n)$ denote the probability of obtaining a cumulative sum of exactly n in repeated rolls of a *single* die. The derivation which follows uses standard combinatorial techniques which can be found in most undergraduate combinatorics and probability textbooks, *e.g.*, Section 1.6 of *A First Course in Probability* by Sheldon Ross.

The probability of obtaining a *particular* sequence of k rolls with sum n is $(1/6)^k$. To find the probability that *any* sequence of k rolls sums to n, we find the number of such sequences, and multiply the result by $(1/6)^k$.

We count sequences of rolls using the Principle of Inclusion and Exclusion (PIE). Let S be the set of all solutions to $x_1 + x_2 + \ldots + x_k = n$ in integers $x_i \geq 1, 1 \leq i \leq k$. We want the subset of S consisting of all solutions with $x_i \leq 6$, $1 \leq i \leq k$. Let A_i denote the subset of S containing all solutions with $x_i > 6$. Then $\cup_{i=1}^k A_i$ is the set of solutions we *don't* want, and $S \setminus \left(\cup_{i=1}^k A_i\right)$ is the set of solutions we *do* want. The size of this set, $|S \setminus \left(\cup_{i=1}^k A_i\right)|$, is the number of sequences of k rolls which sum to n.

Now $|S \setminus \left(\cup_{i=1}^k A_i\right)| = |S| - |\cup_{i=1}^k A_i|$, and we can use the PIE to compute the size of the union once we have computed $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_l}|$ for all subsets $\{i_1, i_2, \ldots, i_l\}$ of $\{1, 2, \ldots, k\}$, $1 \leq l \leq k$.

We begin with $|S|$. How many integer solutions are there to the equation $x_1 + x_2 + \cdots + x_k = n$ with $x_i \geq 1, 1 \leq i \leq k$? This is equivalent to the number of ways to separate n identical balls into k distinct boxes so that each box contains at least one ball (box i contains x_i balls). If we let $x_i = 1 + y_i$ (place one ball into each box to start), then $|S|$ is also the number of ways to separate $n - k$ balls into k boxes with no restriction (box i contains y_i balls in addition to the one we started with). To count such distributions, we view a particular arrangement of balls in boxes as a straight line of balls and dividers. We line up all the balls from the first box, then place a divider in the line to separate them from the balls in the second box; we line up the balls from the second box following the first divider, then place a second divider in the line to separate them from the balls in the third box, and so on. All together we have $n-k$ balls and $k - 1$ dividers. The number of balls in the line between the $(i-1)^{st}$ and i^{th} dividers is y_i, the number of additional balls added to box i after the first. Thus $|S|$ is the number of linear arrangements of $n - k$ identical balls and $k - 1$ identical dividers. Of the $n - 1 = (n - k) + (k - 1)$ positions in line, we choose $k-1$ for the dividers, and fill in the remaining with balls. There are $\binom{n-1}{k-1}$ ways to do this. This means there are $\binom{n-1}{k-1}$ solutions to $y_1 + y_2 + \ldots + y_k = n - k$ in nonnegative integers, and therefore $\binom{n-1}{k-1}$ solutions $x_1 + x_2 + \ldots + x_k = n$ in positive integers. We have shown that $|S| = \binom{n-1}{k-1}$.

In a similar way we can find $|A_i|$, the number of solutions with $x_i > 6$, $|A_i \cap A_j|$, the number of solutions with $x_i > 6$ and $x_j > 6$, and so on. To do the latter we start with 7 ball in boxes i and j ($x_i = 7 + y_i$ and $x_j = 7 + y_j$) and 1 ball in each of the other boxes. This leaves $n - k - 6(2)$ balls to distribute among k boxes without restriction. We can do this in $\binom{n-6(2)-1}{k-1}$ ways. Since there are $\binom{k}{2}$ choices for i and j, we have $\binom{k}{2}$ intersections of the form $A_i \cap A_j$ to consider. With these computations completed, we compute $|S| - |\cup_{i=1}^{k} A_i|$ as

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-6i-1}{k-1}.$$

This is the number of sequences of k rolls of a single die that sum to n. The probability that a sequence of k rolls sums to n is therefore

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-6i-1}{k-1} \left(\frac{1}{6}\right)^k.$$

We sum over k to find the probability that a sequence of any number of rolls has a cumulative sum of n. We obtain

$$P(n) = \sum_{k \geq 1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-6i-1}{k-1} \left(\frac{1}{6}\right)^k.$$

6 Extensions.

Typically one can obtain a particular mathematical result in a variety of different ways, and must choose the "best" one to present. The path to the formula for $P(n)$ described above includes the author's favorite kind of combinatorics, though she acknowledges it may be longer and less direct than other approaches. She challenges the reader to derive the same result from the partial fraction decomposition of $F(z) - J(z)$ in the previous section, or the recurrence $P(n) = \frac{1}{6}P(n-1) + \frac{1}{6}P(n-2) + \frac{1}{6}P(n-3) + \frac{1}{6}P(n-4) + \frac{1}{6}P(n-5) + \frac{1}{6}P(n-6)$ for $n \geq 6$.

Rollette is just one example of a game based on repeated dice rolls. The generating function $F(z)$ can also be used to compute probabilities for endless variations on this game. In addition one might consider games with more than one player, more than two dice, or loaded dice. How does the game change, for example, when the probability p_i of adding i to a cumulative sum is based on a process other than the roll of two dice (*e.g.* a spin of a *Roulette* wheel)? What is $F(z)$ in this case?

About the author:

The author completed her Ph.D. in 1999 at Dartmouth under the supervision of Ken Bogart. She held positions at the University of Pennsylvania and St. Joseph's University in Philadelphia before accepting her current appointment at Bryn Mawr College.

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