Shellability of Interval Orders *

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Abstract

An finite interval order is a partially ordered set whose elements are in correspondence with a finite set of intervals in the line, with disjoint intervals being ordered by their relative position. We show that any such order is shellable in the sense that its (not necessarily pure) order complex is shellable.

1 Introduction.

The study of finite posets by means of the topological properties of their order complexes is a standard technique in enumerative combinatorics (see, *e.g.*, [7, §3.8]). In the usual case, these posets are ranked (i.e., maximal chains all have the same length) and so the order complex has pure dimension (i.e., maximal simplices all have the same dimension), and it was reasonable to ask whether further conditions of a geometric nature would hold for these complexes. Such conditions included shellability and the property of being Cohen-Macaulay, both of which included pure dimensionality, either as part of the definition (as in the former case) or as a direct corollary (as in the latter). (See, *e.g.* [8, Chapter III] or [3].) Since interval orders are not usually ranked, their study in this framework did not seem to hold much interest.

However, recently the study of shellability of complexes has been extended by Björner and Wachs to include non-pure complexes [4, 5], and so it becomes reasonable to ask whether interval orders are shellable in this non-pure sense. We show here that they always are.

We begin with a few definitions. Given a partially ordered set P, the *order complex* $\Delta(P)$ of P is the simplicial complex whose vertices are the elements of P and whose simplices are the chains in P. Equivalently, $\Delta(P)$ is the order ideal of subsets of P consisting of its totally ordered subsets. Thus maximal simplices in $\Delta(P)$ correspond to the maximal chains of P.

A (not necessarily pure) finite simplicial complex is said to be *shellable* if there is an ordering of its maximal simplices $F_1, F_2, \dots, F_t$ so that for each $k = 2, \dots, t$, the complex $\overline{F}_k \cap (\overline{F}_1 \cup \cdots \cup \overline{F}_{k-1})$ is pure of dimension

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$|F_k| - 2$. Here, for $F \in \Delta$, $\overline{F}$ is the complex consisting of F and all its subsets. Such an ordering is called a *shelling* of Δ. We say a finite partially ordered set is shellable if its order complex is a shellable simplicial complex. See [3, 4, 5] for more information on shellings of posets, in particular, non-pure shellings.

An interval order $\mathcal{I}$ is an ordered set with the property that each element $x \in \mathcal{I}$ can be assigned to a closed interval $I(x)$ of real numbers so that $x < y$ in $\mathcal{I}$ if and only if every point in $I(x)$ is less than every point in $I(y)$. In other words $x < y$ in $\mathcal{I}$ if and only if the interval $I(x)$ lies completely to the left of the interval $I(y)$ on the real number line. Equivalently, a partially ordered set $\mathcal{I}$ is an interval order if and only if for any $a, b, c, d \in \mathcal{I}$ satisfying $a < b$ and $c < d$, we also have either $a < d$ or $c < b$ or both (see, *e.g.* [6, Chapter 2]).

Interval orders arose as generalizations of semiorders. A semiorder is an ordered set S with the same property satisfied by interval orders: for $a, b, c, d \in S$ with $a < b$ and $c < d$, we also have either $a < d$ or $c < b$ or both; and the additional property: for $a, b, c, d \in S$ with $a < b$ and $b < c$, we also have either $a < d$ or $d < c$ or both. Here each $x \in S$ can be assigned to a closed interval $I(x)$ as above where $I(x)$ has a specified length independent of x. Equivalently S is a semiorder if and only if each element $x \in S$ can be assigned a real number $r(x)$ such that $x < y$ in S if and only if $r(y)$ is "sufficiently larger" than $r(x)$. Specifically, for some given δ independent of x and y we have $r(x) + \delta \leq r(y)$. With this definition we can view y as being "preferred" to x by some decision maker if $r(y)$ is noticeably larger than $r(x)$, otherwise the decision maker is indifferent about the choice between y and x. It was in the context of preferences that semiorders were originally studied. For more on interval orders and their applications see [6, 9].

The main result of this note is the following.

Theorem 1 *Finite interval orders are shellable.*

Since we announced this result, alternate proofs have been offered making use of the techniques of lexicographic shellability (for example [10, Theorem 9]). The proof we give here is entirely elementary and self-contained.

2 The shelling.

Let $\mathcal{I}$ be an interval order. For an interval $I \in \mathcal{I}$, denote the predecessor and successor sets of I by $pred(I) = \{J \in \mathcal{I} | J < I\}$ and $suc(I) = \{J \in \mathcal{I} | I < J\}$. Let $\ell : \mathcal{I} \to \{1, 2, \cdots, |\mathcal{I}|\}$ be a bijection satisfying $\ell(I) < \ell(J)$ if $pred(I) \subset pred(J)$ or $pred(I) = pred(J)$ and $suc(I) \supset suc(J)$. (If both $pred(I) = pred(J)$ and $suc(I) = suc(J)$, then the relative order of $\ell(I)$ and $\ell(J)$ is arbitrary.)

Totally order the collection $\mathbf{C}(\mathcal{I})$ of maximal chains in $\mathcal{I}$ lexicographically as follows. For $C, D \in \mathbf{C}(\mathcal{I})$, we say $C < D$ if $|C| > |D|$, that is if the number of intervals in the chain C is greater than the number of intervals in D, or if $|C| = |D|$ and the minimum label of an interval in $C \backslash D$ is larger than the minimum label of an interval in $D \backslash C$.

To prove Theorem 1, we will establish the following.

Theorem 2 *The linear order on the maximal chains of $\mathcal{I}$ is a shelling of $\mathcal{I}$.*

Proof: To show that the linear order on $\mathbf{C}(\mathcal{I})$ described above gives a shelling of $\mathcal{I}$, we must demonstrate the following:

> (†) For all C and D in $\mathbf{C}(\mathcal{I})$ with $C < D$, there exists an E in $\mathbf{C}(\mathcal{I})$ such that $E < D$, $D \cap C \subseteq D \cap E$, and $|D \cap E| = |D| - 1$.

To prove (†), suppose C and D are chains in $\mathbf{C}(\mathcal{I})$ with $C < D$. If $|D \cap C| = |D| - 1$, then let $E = C$. If $|D \cap C| < |D| - 1$, then consider the two intervals I and J in D such that I and J are not in C and $\ell(I)$ and $\ell(J)$ are minimal. We have the following situation.

$$C = C_1 < K < C_2 < L < C_3$$
$$D = C_1 < I < C_2 < J < D_3$$

where C_1, C_2, C_3, and D_3 are (possibly empty) chains of intervals, and I, J, K, and L are intervals.

Since $C < D$ in $\mathbf{C}(I)$ we know either $\ell(K) > \ell(I)$, $|C_3| > |D_3|$ or both. If $C_2 \neq \emptyset$ and $\ell(K) > \ell(I)$, then let $E = C_1 < K < C_2 < J < D_3$. If $C_2 \neq \emptyset$ and $\ell(K) < \ell(I)$, then we must have $|C_3| > |D_3|$. In this case set $C' = C_1 < I < C_2 < L < C_3$ and repeat this process with C' and D. Note that $|D \cap C'| = |D \cap C| + 1$.

If $C_2 = \emptyset$, then we consider the following two cases.

Case 1. $\ell(K) > \ell(I)$

If $K < J$, then let E be any maximal chain in $\mathcal{I}$ containing the chain $C_1 < K < J < D_3$. If K overlaps J, then we must have $I < L$ and $pred(J) \subset pred(L)$, and so $\ell(L) > \ell(J)$. Letting C' be a maximal chain containing $C_1 < I < L < C_3$, we have either $|C'| > |D|$ or the first place where they differ is on J and L. In either case, we conclude $C' < D$, and we can repeat this process with C' and D. Note that $|D \cap C'| = |D \cap C| + 1$.

Case 2. $\ell(K) < \ell(I)$

Here we must have $|C_3| > |D_3|$. If $L < J$, then let E be any maximal chain in $\mathcal{I}$ containing the chain $C_1 < K < L < J < D_3$. If $I < L$, then let C' be a maximal chain containing the chain $C_1 < I < L < C_3$. Here $|C'| > |D|$ and so $C' < D$, and we can repeat this process with C' and D. Again $|D \cap C'| = |D \cap C| + 1$.

If L overlaps both I and J, then let C' be any maximal chain in $\mathcal{I}$ containing the chain $C_1 < I < C_3$. Since $|C_3| > |D_3|$ we know $|C'| \geq |D|$. If $|C'| = |D|$, then the fact that L overlaps J means that any interval M in C_3 satisfies $pred(J) \subset pred(M)$, whereas $L \in pred(M)$ and $L \notin pred(J)$. Thus $\ell(M) > \ell(J)$ and so in either case $C' < D$. Again, we can repeat this process with C' and D; as before, $|D \cap C'| = |D \cap C| + 1$. $\diamond$

3 Further comments and examples.

We note here two direct corollaries of Theorem 1.

It is a characterization of interval orders that they are precisely those posets not having as an induced subposet the 4-element poset Q consisting of disjoint 2-element chains. It is easy to see that Q is not shellable. Wachs [10] pointed out that Theorem 1 is equivalent to the following

Corollary 1 *Every nonshellable poset contains Q as an induced subposet.*

In this sense, Q is a *witness* to nonshellability of posets, in that it must be present in a nonshellable poset. (However, it is not an entirely reliable witness in that it may also appear as an induced subposet of a shellable poset, for example, the lattice of subsets of a 3-element set.) This suggests a similar study for other classes of simplicial complexes, the first steps of which are carried out in [10].

A different application of Theorem 1 to the study of sparse subsets of $\{1, 2, \ldots, n\}$ was suggested by Ehrenborg (personal communication). These arise in the study of chain enumeration in Eulerian posets (see [1, 2]).

A subset $S \subset \{1, 2, \ldots, n\}$ is said to be *sparse* if it contains no two consecutive elements $i, i + 1$. The collection of all sparse subsets of $S \subset \{1, 2, \ldots, n\}$ forms a simplicial complex, which is not pure if n is odd. For example, the maximal sparse subsets of $\{1, \ldots, 5\}$ are $\{1, 3, 5\}$, $\{1, 4\}$, $\{2, 4\}$ and $\{2, 5\}$. Nonetheless, the complex of sparse sets is always shellable.

Corollary 2 *The simplicial complex of sparse subsets of* $\{1, 2, \ldots, n\}$ *is always shellable.*

Proof: It is easy to see that the complex of sparse subsets of $\{1, 2, \ldots, n\}$ is isomorphic to the order complex of the interval order on the set of intervals $\{\ [i, i + 1] \mid 1 \leq i \leq n\ \}$. $\diamond$

Finally, recall that a composition of an integer $N > 0$ is a sequence $a_1, \ldots, a_k$ of positive integers such that $a_1 + \cdots + a_k = N$. If we define $\mathcal{I}[0, n]$ to be the interval order realized by the multiset of *all* intervals in $[0, n]$ with integer endpoints, we note that maximal chains in $\mathcal{I}[0, n]$ correspond bijectively to compositions of $n + 1$ via the map $I_1 < I_2 < \cdots < I_p \mapsto (|I_1|, |I_2|, \cdots, |I_p|)$. The shelling order on $\mathcal{I}[0, n]$ agrees, under this bijection, with the lexicographic order on compositions.

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