Are Venn Diagrams Limited to Three or Fewer Sets?

Amy N. Myers

As teachers we introduce Venn diagrams to provide students with a useful tool for thinking about sets, logic, counting, probability, and more. Typically we draw Venn diagrams to visualize the intersections among two or three sets. Why do we stop at three? Can we find no interesting problems whose analyses require the consideration of four or more sets? Why wouldn't we like the class to consider, for example, the students who are athletes, artists, student government leaders, and/or academic team members, using a Venn diagram to illustrate the intersections among these four sets?

We typically create Venn diagrams using congruent circles as shown in Figure 1. When we try to arrange four congruent circles to represent the intersections among four sets, we quickly realize this task is not as straightforward as it is for fewer sets. (If you haven't already tried to do this, please do so before continuing!) Is there a clever way to arrange the circles, or is such a diagram simply impossible?

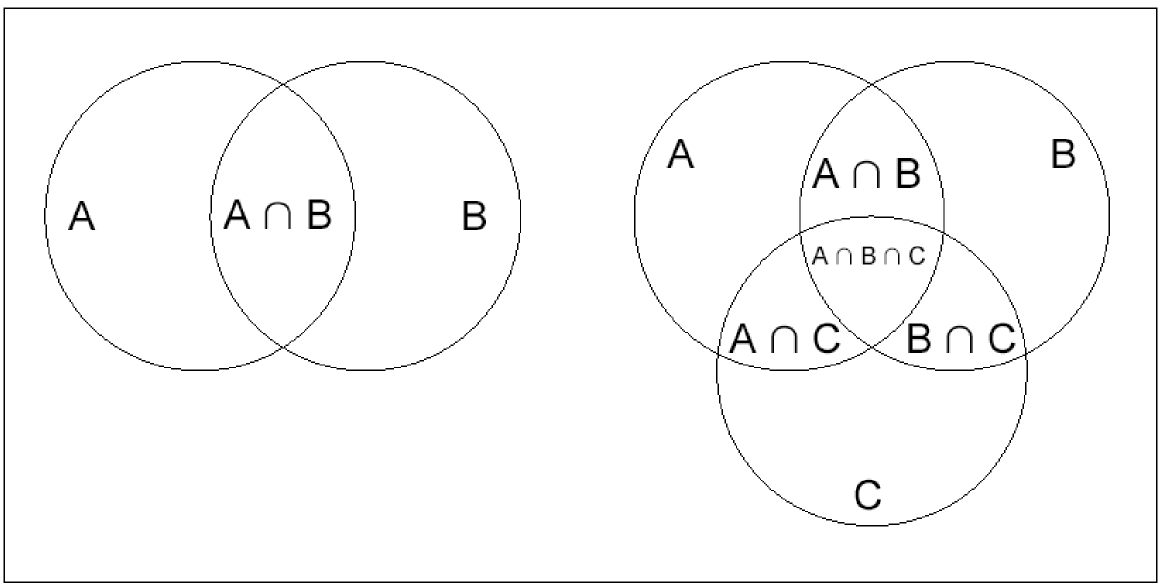

Figure 1. *Venn diagrams illustrating the intersections among 2 and 3 sets..*

After spending some time arranging and rearranging four congruent circles in an attempt to represent all the intersections among four sets, and failing to do so, we may begin to suspect that such a Venn diagram simply does not exist. But surely we ought to be able to produce some sort of diagram that allows us to visualize all the relationships among four sets? If we relax the congruency requirement, could we produce such a diagram?

If four congruent circles cannot be arranged to form a satisfactory Venn diagram, then perhaps we could create the desired diagram using four non-congruent circles. (Try it!) Unfortunately this exercise soon proves frustrating as well. (If you think you've produced all the possible intersections with four circles, congruent or otherwise, please count again! The Venn diagram should show 16 regions, including the one outside the circles.) Are we missing something, or is a Venn diagram using circles for four (or more) sets simply impossible?

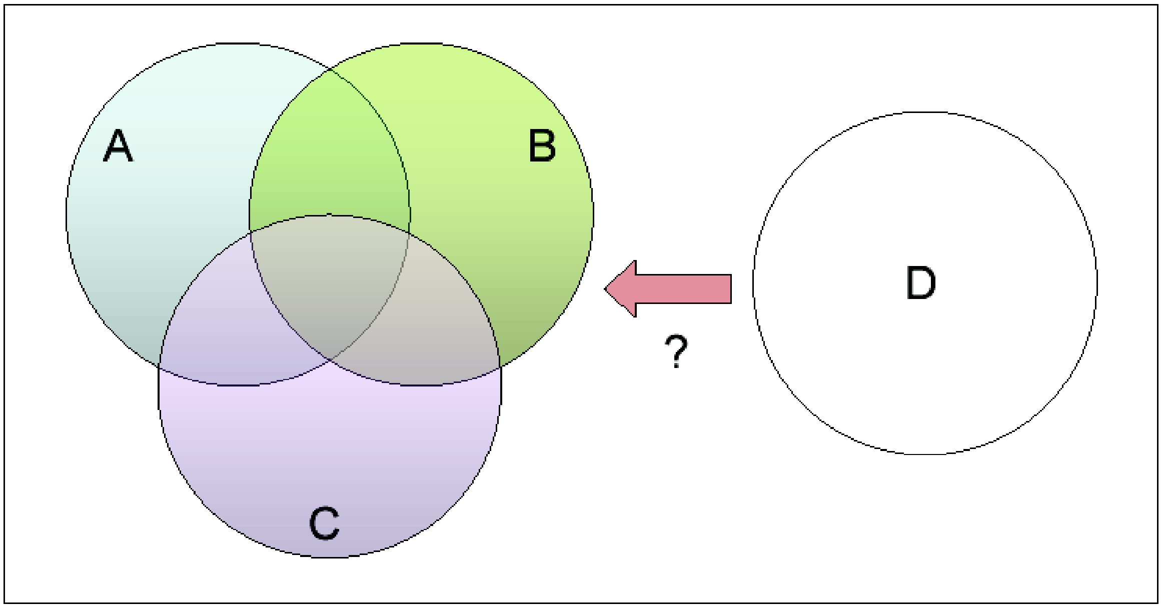

Figure 2. *Can we use four congruent circles to represent all possible intersections among four sets?*

Below we show that a Venn diagram using circles to represent the intersections among $n \geq 4$ sets can't exist. We can, however, produce Venn diagrams for four or more sets using other shapes.

How many regions can we create with n circles? How many regions must a Venn diagram have in order to display all the possible intersections among n sets? We answer the second question first.

Suppose we have a collection of objects (or students), each of which may or may not belong to any one of the n sets (or clubs) $A_1, A_2, \ldots, A_n$. With each object x we associate an n-tuple $(x_1, x_2, \ldots, x_n)$, where $x_i \in \{Y, N\}$ for $1 \leq i \leq n$, and $x_i = Y$ indicates the object is in set A_n and $x_i = N$ means it is not. Each distinct n-tuple requires a distinct region in a Venn diagram and vice versa. This means the number of regions in a diagram that displays all the possible relationships among n sets must equal the number of distinct n-tuples with entries Y or N. How many such n-tuples are there? Since each x_i independently takes one of two possible values, there must be 2^n of them. This means a Venn diagram that displays the intersections among n sets must have exactly 2^n regions. In particular a Venn diagram for 4 sets must have 16 regions. Figure 3 illustrates the eight 3-tuples that correspond to the $2^3 = 8$ regions in a Venn diagram for 3 sets.

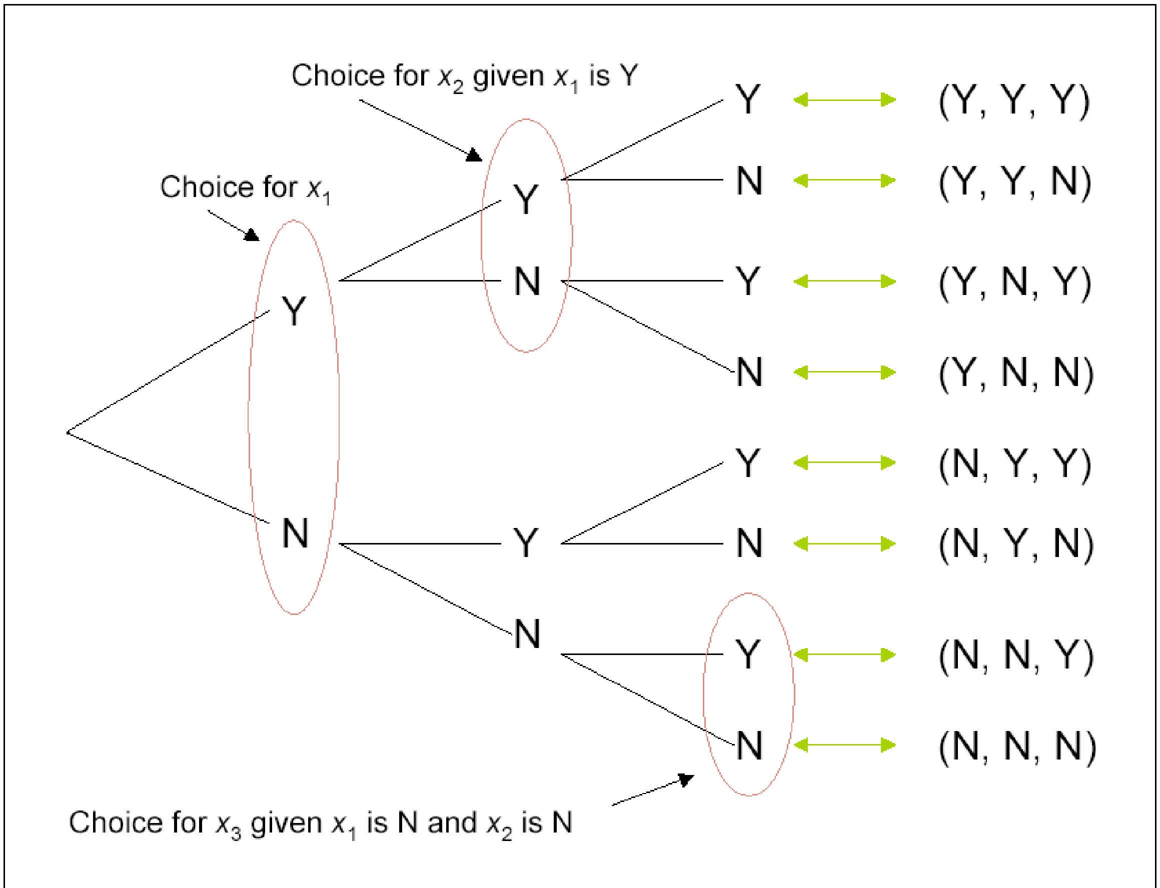

Figure 3. *Two choices for each x_i independently yields $2 \times 2 \times 2 = 2^3$ 3-tuples (x_1, x_2, x_3).*

How many regions can we create by arranging n circles? To maximize the number of regions, we make sure no more than two circles intersect at a given point. Let r_n denote this maximum number.

Now suppose we have $n - 1$ circles drawn already with a total of r_{n-1} regions. How many more regions can the addition of one more circle yield? To maximize the number of regions, we draw the n^{th} circle so that it intersects the existing $n - 1$ circles in two distinct points each (nonintersecting and tangent circles produce no new regions). When the n^{th} circle intersects an existing circle, it creates two new regions: it begins one new region when it enters the existing circle, and starts another upon leaving the circle. (Try this for a small number of circles to visualize the formation of regions.) See Figure 4 for an illustration of this process.

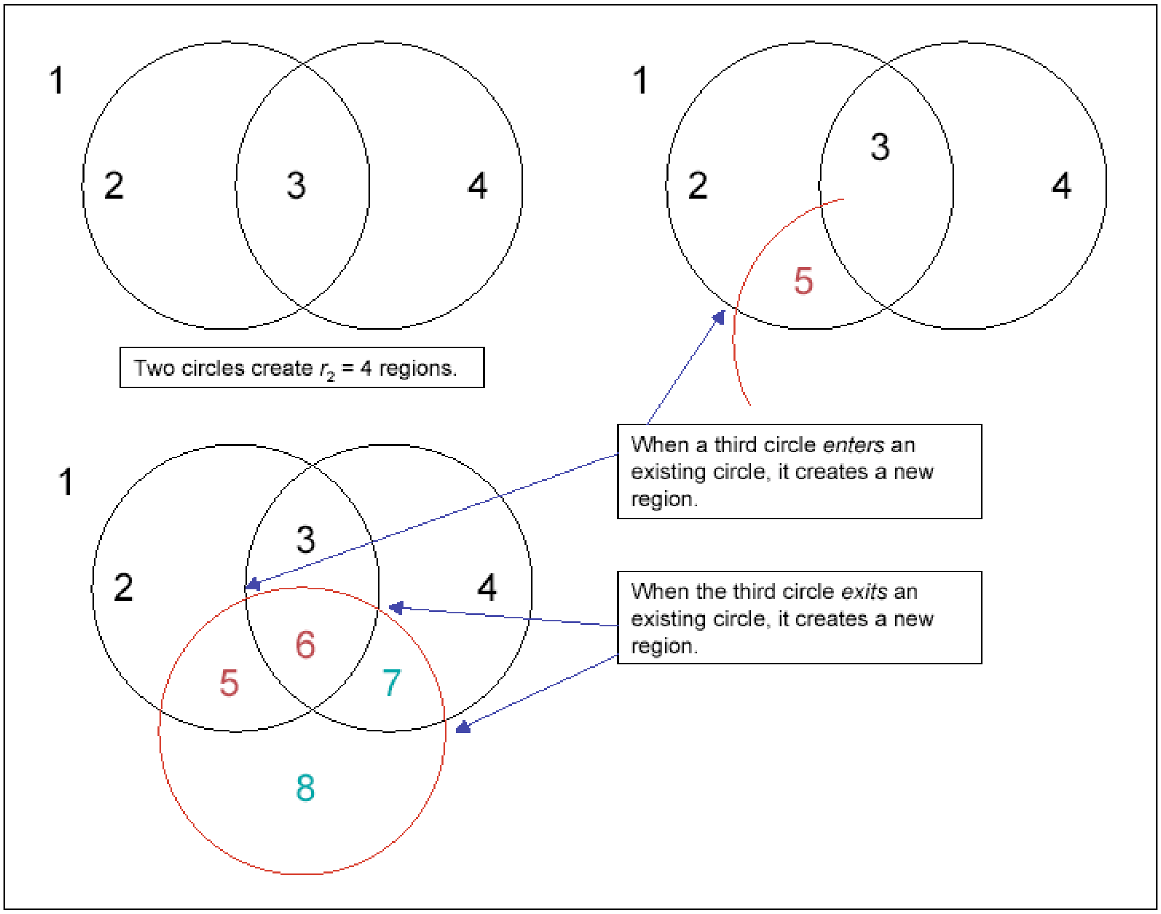

Figure 4. *Adding a circle creates a new region each time the new circle both enters and exists an existing circle.*

The above analysis demonstrates that $r_n = r_{n-1} + 2(n-1)$. To obtain the maximum number r_n of regions that can be created with n circles, we begin with $n - 1$ circles arranged to form as many regions as possible, that is r_{n-1}regions. Then we add the n^{th} circle so that it intersects the existing $n - 1$ circles in two places each. Every time the new circle enters or exists an existing circle, it creates a new region. Since this amounts to 2 new regions for each of the $n - 1$ existing circles, the total number of new regions added by the n^{th} circle is $2(n - 1)$. Combining these new regions together with the r_{n-1} existing regions yields the result.

Since the recurrence $r_n = r_{n-1} + 2(n-1)$ holds for $n \geq 2$ and $r_1 = 2$, we can obtain r_n for n as large as we have the patience to compute (see Figure 5).

n	r_n
1	2
2	$4 = 2 + 2(1)$
3	$8 = 4 + 2(2)$
4	$14 = 8 + 2(3)$
5	$22 = 14 + 2(4)$
n	$r_n = r_{n-1} + 2(n - 1)$

Figure 5. *Computing the maximum number r_n of regions formed by n circles using the recurrence relation.*

Is there an explicit formula for r_n? For large values of n, it would be nice to have a formula for r_n that depends only on n, so that we don't have to rely on the recurrence. The substitutions and computations shown in Figure 6 demonstrate that $r_n = n^2 - n + 2$.

$$
\begin{aligned}
r_n &= r_{n-1} + 2(n - 1) \\
&= r_{n-2} + 2(n - 2) + 2(n - 1) \\
&= r_{n-3} + 2(n - 3) + 2(n - 2) + 2(n - 1) \\
&= \ldots \\
&= r_1 + 2(1) + 2(2) + 2(3) + \ldots + 2(n - 3) + 2(n - 2) + 2(n - 1) \\
&= 2 + 2(1) + 2(2) + 2(3) + \ldots + 2(n - 3) + 2(n - 2) + 2(n - 1) \\
&= 2 + 2(1 + 2 + 3 + (n - 3) + (n - 2) + (n - 1)) \\
&= 2 + 2(n(n - 1) / 2) \\
&= 2 + n(n - 1) \\
&= n^2 - n + 2
\end{aligned}
$$

Figure 6. *An explicit formula for the maximum number r_n of regions formed by n circles.*

We see that $n^2 - n + 2 = 2^n$ for $n = 1$, 2, and 3, but $n^2 - n + 2 < 2^n$ for $n \geq 4$. We need 2^n regions in a Venn diagram for n sets, but can create at most $n^2 - n + 2$ regions from the intersection of n circles. This means we can construct Venn diagrams using circles only for three or fewer sets.

Suppose we need a Venn diagram for 4 sets. We know we cannot use circles, congruent or otherwise. Figure 7 provides one possibility, but it is aesthetically unsatisfying. The shapes are not congruent, nor are they convex, and the diagram does not have the rotational symmetry shared by the Venn diagrams in Figure 1. Are these qualities too much to ask for? Can we produce a Venn diagram using congruent shapes? Can we ask that they be convex, or that the diagram be symmetric? Figure 4 guarantees that a Venn diagram for 4 sets is possible, but can we create one with more appealing qualities?

More generally, do Venn diagrams exist for all n? What kinds of visually desirable characteristics can we hope to achieve?

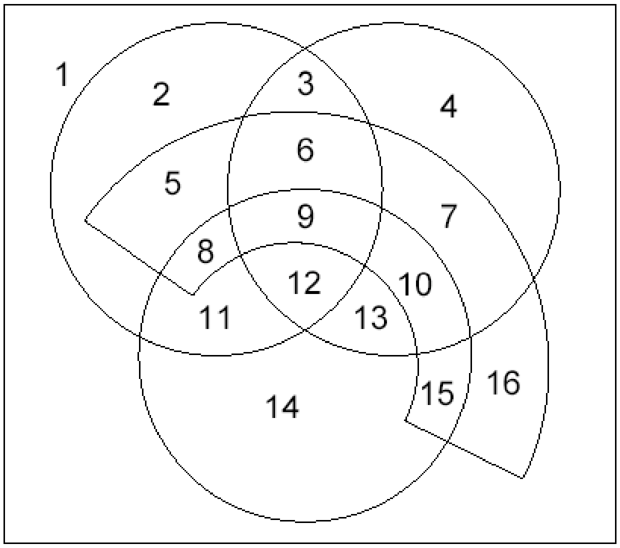

Figure 7. *A Venn diagram for 4 sets using non-congruent shapes.*

When we search the mathematical literature for answers to these questions, we discover that Venn diagrams have inspired a lot of mathematical research. Among the results we find partial answers to our questions, together with several loose ends that remain to be tied by future mathematicians. (Will our students be among them?) The online article [7] contains a survey of the current state of mathematical research concerning Venn diagrams. It includes several illustrations, and lots of open problems.

Venn himself showed that Venn diagrams exist for all n. He did so by adding more shapes to Figure 7 in a systematic (though increasingly complex) way. His diagrams used neither congruent, nor convex shapes, and had no rotational symmetry. These aesthetic considerations did not trouble him, however, since he felt their value depended not on their appearance, but rather on their purpose [9].

Since mathematical problems can be solved in a variety of ways, why not seek the nicest solution? A simple diagram is easier to use than a complex one. What could be simpler than a diagram with congruent, convex shapes arranged symmetrically? Figure 8 provides such a diagram for 5 sets.

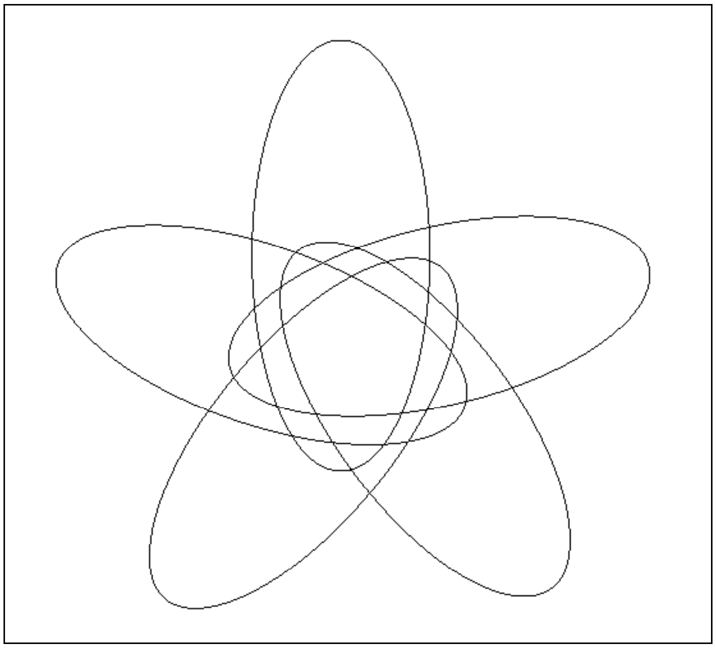

Figure 8. *A simple, symmetric Venn diagram for 5 sets using convex, congruent ellipses.*

The search for ideal Venn diagrams eventually forces one to think carefully about which arrangements of *n* shapes should qualify as "simple" and "symmetric". A Venn diagram in which no more than two shapes intersect at a given point is called *simple*. Figures 1, 7 and 8 display simple Venn diagrams. Figures 9 and 10 are non-simple Venn diagrams for 3 sets. To say a Venn diagram for *n* sets is *symmetric* means it is has *n*-fold rotational symmetry. Figure 8 has 5-fold rotational symmetry, while Figure 9 has 3-fold rotational symmetry. The Venn diagrams in Figure 1 are simple and symmetric.

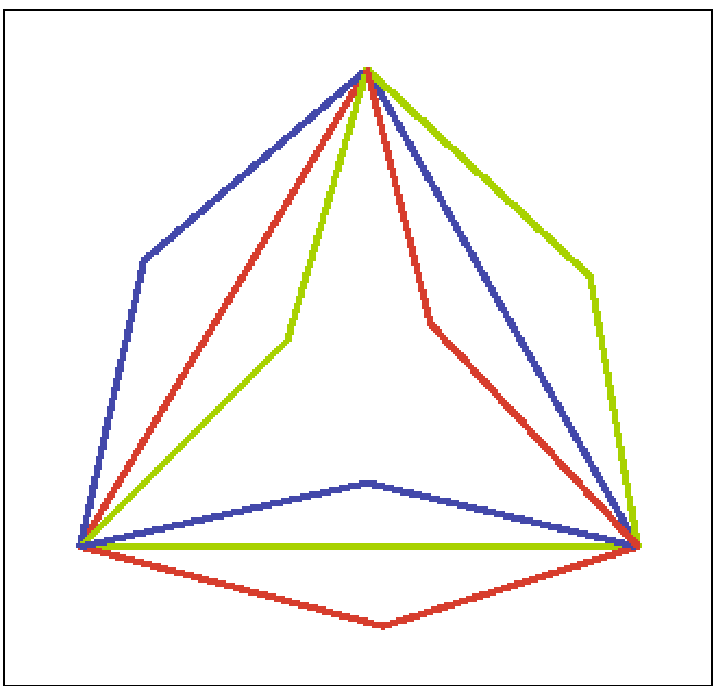

Figure 9. *A non-simple, symmetric Venn diagram for 3 sets using congruent, non-convex pentagons.*

If we allow intersections along curves, then we obtain diagrams such as the one in Figure 10 (which was copied from [7]). The shapes composing this diagram are congruent and convex, but the diagram is neither symmetric nor simple.

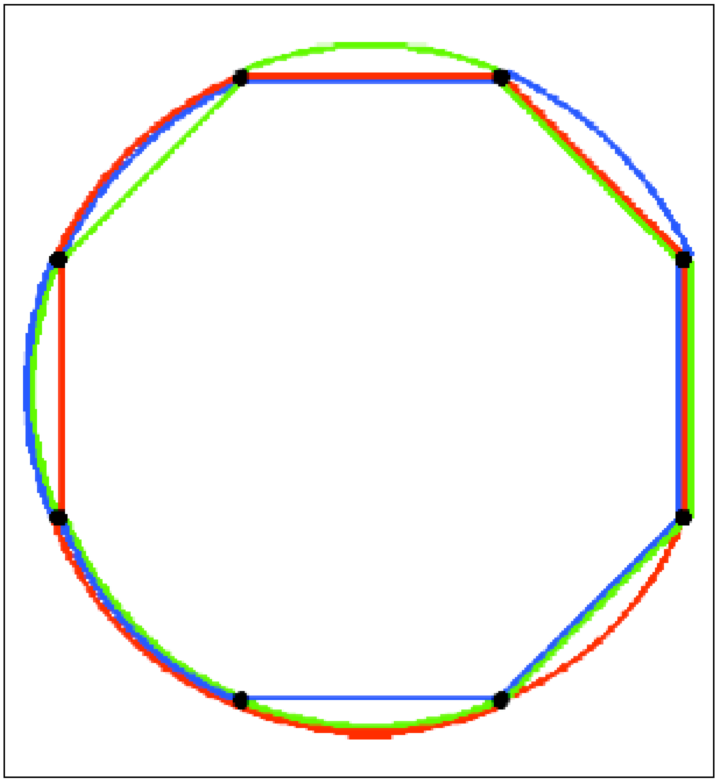

Figure 10. *A non-simple, non-symmetric Venn diagram for 3 sets using congruent, convex shapes.*

The ideal Venn diagram would be simple, symmetric, and consist of congruent, convex shapes, as is the case for 1, 2, 3, and 5 sets (see Figures 1 and 8). Figures 9 and 10 suggest we can have several desirable properties simultaneously for larger n. Having all of them at once, however, is unfortunately too much to ask for. Below we describe the current state of the search for ideal Venn diagrams.

Symmetric Venn diagrams for n sets exist only for prime n. A nice explanation for why this is true appears in [6]. (For the original articles see [2] and [5].) *Simple* symmetric Venn diagrams have been constructed only for $n = 1, 2, 3, 5$ and 7. The problem of finding simple symmetric Venn diagrams for prime $n \geq 11$ is open.

Venn diagrams using convex shapes can be produced for any n, but they will not necessarily be congruent, unless we allow Venn diagrams to use shapes that intersect along curves, such as those in Figure 10, (see [1]). Diagrams using congruent (though not necessarily convex) shapes that intersect at only finitely many points (*i.e.*, not along curves) have been constructed only for $n \leq 8$ (see [3] and [4]). The problem of construction for $n \geq 9$ is open. Figure 11 displays one such diagram for 4 sets created by Venn himself (see [8]). It is simple, but not symmetric. The problem of constructing Venn diagrams for n sets using *congruent* shapes for non-prime $n \geq 6$ is also open.

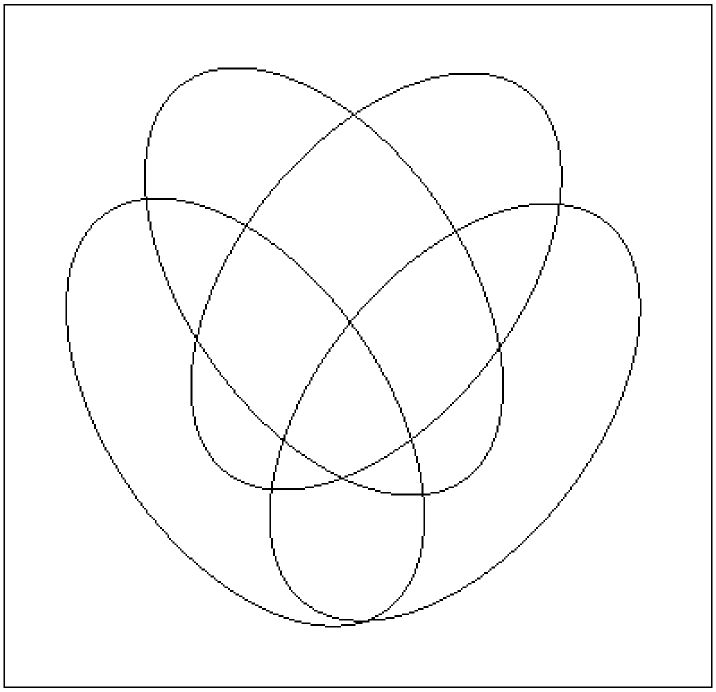

Figure 11. *A simple, non-symmetric Venn diagram for 4 sets using congruent, convex ellipses.*

Perhaps the real reason we don't consider Venn diagrams for four or more sets is that the diagrams (even ideal ones) become increasing difficult to manage (not to mention draw). The number 2^n of regions becomes very large quite quickly. The question of producing ideal Venn diagrams, does, however, lead to lots of interesting and beautiful mathematics, much of which remains to be created by future generations of researchers. The open problems described here represent only a small fraction of those that appear as part of the excellent online survey article [7], to which the reader is (enthusiastically!) referred.

References

[1] Ronald Graham, Donald Knuth, and Oren Patashnik, Concrete Mathematics, Addison-Wesley, (1989).

[2] Jerrold Griggs, Charles Killian, and Carla Savage, Venn diagrams and symmetric chain decompositions in the Boolean lattice, *Electronic Journal of Combinatorics*, 11 no. 1 (2004), R2.

[3] Branko Grünbaum, Venn diagrams and independent families of sets, *Mathematics Magazine*, 48 Jan – Feb (1975), 12 – 23.

[4] Branko Grünbaum, Venn Diagrams II, *Geombinatorics*, 1 no. 2 (1992), 25 – 32.

[5] David Henderson, Venn diagrams for more than four classes, *American Mathematical Monthly*, 70 (1963), 424 – 426.

[6] Frank Ruskey, Carla Savage, and Stan Wagon, The search for simple symmetric Venn diagrams, *Notices of the American Mathematical Society*, 53 no. 11 (2006), 1304 – 1312. Available at http://www.ams.org/notices/200611/fea-wagon.pdf

[7] Frank Ruskey and Mark Weston, A survey of Venn diagrams, *Electronic Journal of Combinatorics*, 4 (1997), DS5, (updated 2001, 2005). Available at http://www.combinatorics.org/Surveys/ds5/VennEJC.html

[8] John Venn, On the diagrammatic and mechanical representation of propositions and reasonings, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 9 (1880) 1 – 18.

[9] John Venn, Symbolic Logic, MacMillan, (1894).