CONNECTED SETS, INTERVALS AND CONTINUOUS FUNCTIONS

Theorem: An interval is connected.

Proof: (from Morgan, Ch. 12, p. 49) Suppose that an interval I can be separated by two disjoint open sets into two non-empty pieces $I \cap U_1$ and $I \cap U_2$. Take a point $a_1 \in I \cap U_1$ and a point $a_2 \in I \cap U_2$. We may suppose that $a_1 < a_2$.

Since I is an interval, the entire interval $[a_1, a_2]$ is contained in I and hence is covered by the open sets U_1 and U_2. So each point in the interval $[a_1, a_2]$ is an element of either U_1 or U_2.

Consider the set $S_1 = [a_1, a_2] \setminus U_2 = [a_1, a_2] \cap U_2^c$. Let $b_1 =$ maximum element of S_1, which exists since S_1 is compact and non-empty (since $a_1 \in S_1$). Then $b_1 \in U_1$ (since it is not an element of U_2) and thus $b_1 < a_2$.

Now examine the set $S_2 = [b_1, a_2] \setminus U_1 = [b_1, a_2] \cap U_1^c$. Let $b_2 =$ minimum element of S_2. Then $b_2 \in U_2$. $b_2 \geq b_1$. Since $b_1 \in U_1$, b_1 can not be this minimal element so $b_2 > b_1$.

Choose b_3 so that $b_1 < b_3 < b_2$. Since b_2 is the smallest number larger than b_1 that is not in U_1, the number b_3 must be in U_1.

Then $b_3 \notin U_2$ which contradicts the choice of b_1.

Theorem: (from Morgan, Ch. 12, p. 50) The continuous image of a connected set is connected: i.e. if S is connected then $f(S)$ is connected.

Proof Suppose $f(S)$ is disconnected. Then there exist disjoint open sets U_1 and U_2 that disconnect $f(S)$. Then the sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open, since f is continuous and we claim that these open sets disconnect S.

One must check that

i. $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint

ii. $S \cap f^{-1}(U_1)$ and $S \cap f^{-1}(U_2)$ are non-empty

iii. $S = (S \cap f^{-1}(U_1)) \cup (S \cap f^{-1}(U_2))$

These conditions can all be checked by taking $s \in S$, applying $f(s)$ and using the properties related to U_1 and U_2.

As an example, we prove (i.) Suppose $x \in f^{-1}(U_1) \cap f^{-1}(U_2)$. Then $f(x) \in U_1$ and $f(x) \in U_2$. So $f(x) \in U_1 \cap U_2$ which contradicts that U_1 and U_2 are disjoint.