CONNECTED SETS, INTERVALS AND CONTINUOUS FUNCTIONS

**Theorem:** An interval is connected.

**Proof:** (from Morgan, Ch. 12, p. 49) Suppose that an interval I can be separated by two disjoint open sets into two non-empty pieces \( I \cap U_1 \) and \( I \cap U_2 \). Take a point \( a_1 \in I \cap U_1 \) and a point \( a_2 \in I \cap U_2 \). We may suppose that \( a_1 < a_2 \).

Since \( I \) is an interval, the entire interval \( [a_1, a_2] \) is contained in \( I \) and hence is covered by the open sets \( U_1 \) and \( U_2 \). So each point in the interval \( [a_1, a_2] \) is an element of either \( U_1 \) or \( U_2 \).

Consider the set \( S_1 = [a_1, a_2] \setminus U_2 = [a_1, a_2] \cap U_2^c \). Let \( b_1 = \) maximum element of \( S_1 \), which exists since \( S_1 \) is compact and non-empty (since \( a_1 \in S_1 \)). Then \( b_1 \in U_1 \) (since it is not an element of \( U_2 \)) and thus \( b_1 < a_2 \).

Now examine the set \( S_2 = [b_1, a_2] \setminus U_1 = [b_1, a_2] \cap U_1^c \). Let \( b_2 = \) minimum element of \( S_2 \). Then \( b_2 \in U_2 \). \( b_2 \geq b_1 \). Since \( b_1 \in U_1 \), \( b_1 \) can not be this minimal element so \( b_2 > b_1 \).

Choose \( b_3 \) so that \( b_1 < b_3 < b_2 \). Since \( b_2 \) is the smallest number larger than \( b_1 \) that is not in \( U_1 \), the number \( b_3 \) must be in \( U_1 \).

Then \( b_3 \notin U_2 \) which contradicts the choice of \( b_1 \).

**Theorem:** (from Morgan, Ch. 12, p. 50) The continuous image of a connected set is connected: i.e. if \( S \) is connected then \( f(S) \) is connected.

**Proof** Suppose \( f(S) \) is disconnected. Then there exist disjoint open sets \( U_1 \) and \( U_2 \) that disconnect \( f(S) \). Then the sets \( f^{-1}(U_1) \) and \( f^{-1}(U_2) \) are open, since \( f \) is continuous and we claim that these open sets disconnect \( S \).

One must check that

i. \( f^{-1}(U_1) \) and \( f^{-1}(U_2) \) are disjoint
ii. \( S \cap f^{-1}(U_1) \) and \( S \cap f^{-1}(U_2) \) are non-empty
iii. \( S = (S \cap f^{-1}(U_1)) \cup (S \cap f^{-1}(U_2)) \)

These conditions can all be checked by taking \( s \in S \), applying \( f(s) \) and using the properties related to \( U_1 \) and \( U_2 \).

As an example, we prove (i.) Suppose \( x \in f^{-1}(U_1) \cap f^{-1}(U_2) \). Then \( f(x) \in U_1 \) and \( f(x) \in U_2 \). So \( f(x) \in U_1 \cap U_2 \) which contradicts that \( U_1 \) and \( U_2 \) are disjoint.