Definition. Let $V$ be a set on which the operations of addition and scalar multiplication are defined. By this we mean that with each pair of elements $x$ and $y$ in $V$, one can associate a unique element $x + y$ that is also in $V$, and with each element $x$ in $V$ and each scalar $\alpha$, one can associate a unique element $\alpha x$ in $V$. The set $V$ together with the operations of addition and scalar multiplication is said to form a vector space if the following axioms are satisfied.

A1. $x + y = y + x$ for any $x$ and $y$ in $V$.
A2. $(x + y) + z = x + (y + z)$ for any $x, y, z$ in $V$.
A3. There exists an element $0$ in $V$ such that $x + 0 = x$ for each $x \in V$.
A4. For each $x \in V$, there exists an element $-x$ in $V$ such that $x + (-x) = 0$.
A5. $\alpha(x + y) = \alpha x + \alpha y$ for each real number $\alpha$ and any $x$ and $y$ in $V$.
A6. $(\alpha + \beta)x = \alpha x + \beta x$ for any real numbers $\alpha$ and $\beta$ and any $x \in V$.
A7. $(\alpha \beta)x = \alpha(\beta x)$ for any real numbers $\alpha$ and $\beta$ and any $x \in V$.
A8. $1 \cdot x = x$ for all $x \in V$.

The elements of $V$ are called vectors and are usually denoted by letters from the end of the alphabet: $u, v, w, x, y, z$. The real numbers are called scalars. The symbol $0$ was used in order to distinguish the zero vector from the scalar 0. In some contexts, complex numbers are used for scalars. However, in this book, scalars will usually be real numbers. Often the term real vector space is used to indicate that the set of scalars is the set of real numbers.

An important component of the definition are the closure properties of the two operations. These properties can be summarized as follows:

C1. If $x \in V$ and $\alpha$ is a scalar, then $\alpha x \in V$.
C2. If $x, y \in V$, then $x + y \in V$.

To see the importance of the closure properties, consider the following example. Let

$$W = \{(a, 1) \mid a \text{ real}\}$$

with addition and scalar multiplication defined in the usual way. The elements $(3, 1)$ and $(5, 1)$ are in $W$, but the sum

$$(3, 1) + (5, 1) = (8, 2)$$

is not an element of $W$. The operation $+$ is not really an operation on the set $W$ because property C2 fails to hold. Similarly, scalar multiplication is not
defined on \( W \) because property C1 fails to hold. The set \( W \) together with the operations of addition and scalar multiplication is not a vector space.

On the other hand, if one is given a set \( U \) on which the operations of addition and scalar multiplication have been defined and satisfy properties C1 and C2, one must check to see if the eight axioms are valid in order to determine whether or not \( U \) is a vector space. We leave it to the reader to verify that \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) with the usual addition and scalar multiplication of matrices are both vector spaces. There are a number of other important examples of vector spaces.

THE VECTOR SPACE \( C[a, b] \)

Let \( C[a, b] \) denote the set of all real-valued functions that are defined and continuous on the closed interval \([a, b]\). In this case our universal set is a set of functions. Thus our vectors are the functions in \( C[a, b] \). The sum \( f + g \) of two functions in \( C[a, b] \) is defined by

\[
(f + g)(x) = f(x) + g(x)
\]

for all \( x \) in \([a, b]\). The new function \( f + g \) is an element of \( C[a, b] \), since the sum of two continuous functions is continuous. If \( f \) is a function in \( C[a, b] \) and \( \alpha \) is a real number, define \( \alpha f \) by

\[
(\alpha f)(x) = \alpha f(x)
\]

for all \( x \) in \([a, b]\). Clearly, \( \alpha f \) is in \( C[a, b] \) since a constant times a continuous function is always continuous. Thus on \( C[a, b] \) we have defined the operations of addition and scalar multiplication. To show that the first axiom, \( f + g = g + f \), is satisfied we must show that

\[
(f + g)(x) = (g + f)(x)
\]

for every \( x \) in \([a, b]\). This follows since

\[
(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)
\]

for every \( x \) in \([a, b]\). We leave it to the reader to verify that the remaining vector space axioms are all satisfied.

THE VECTOR SPACE \( P_n \)

Let \( P_n \) denote the set of all polynomials of degree less than \( n \). Define \( p + q \) and \( \alpha p \) by

\[
(p + q)(x) = p(x) + q(x)
\]

and

\[
(\alpha p)(x) = \alpha p(x)
\]

for all real numbers \( x \). It is easily verified that axioms A1 through A8 hold. Thus \( P_n \), with the standard addition and scalar multiplication of functions is a vector space.
ADDITIONAL PROPERTIES OF VECTOR SPACES

We close this section with a theorem that states three more fundamental properties of vector spaces. Other important properties are given in Exercises 7, 8, and 9.

\textbf{Theorem 3.1.1.} If $V$ is a vector space and $x$ is any element of $V$, then

\begin{enumerate}[(i)]
\item $0x = 0$.
\item $x + y = 0$ implies that $y = -x$ (i.e., the additive inverse of $x$ is unique).
\item $(-1)x = -x$.
\end{enumerate}

\textbf{Proof.} It follows from axioms A6 and A8 that

\[ x = 1x = (1 + 0)x = 1x + 0x = x + 0x \]

Thus

\[ -x + x = (x + 0x) = (-x + x) + 0x \quad (A2) \]

\[ 0 = 0 + 0x = 0x \quad (A1, A3, \text{ and } A4) \]

To prove (ii), suppose that $x + y = 0$. Then

\[ -x = -x + 0 = -x + (x + y) \]

Therefore,

\[ -x = (-x + x) + y = 0 + y = y \quad (A1, A2, A3, \text{ and } A4) \]

Finally, to prove (iii), note that

\[ 0 = 0x = (1 + (-1))x = 1x + (-1)x \quad [(i) \text{ and } A6] \]

Thus

\[ x + (-1)x = 0 \quad (A8) \]

and it follows from part (ii) that

\[ (-1)x = -x \]

\[ \square \]

\textbf{EXERCISES}

1. Consider the vectors $x_1 = (8, 6)^T$ and $x_2 = (4, -1)^T$ in $R^2$.

(a) Determine the length of each of the vectors.

(b) Let $x_3 = x_1 + x_2$. Determine the length of $x_3$. How does its length compare to the sum of the lengths of $x_1$ and $x_2$?

(c) Draw a graph illustrating how $x_3$ can be constructed geometrically using $x_1$ and $x_2$. Use this graph to give a geometrical interpretation of your answer to the question in part (b).
Given a vector space $V$, it is often possible to form another vector space by taking a subset $S$ of $V$ and using the operations of $V$. Since $V$ is a vector space, the operations of addition and scalar multiplication always produce another vector in $V$. For a new system using a subset $S$ of $V$ as its universal set to be a vector space, the set $S$ must be closed under the operations of addition and scalar multiplication. That is, the sum of two elements of $S$ must always be an element of $S$ and the product of a scalar and an element of $S$ must always be an element of $S$.

**Definition.** If $S$ is a nonempty subset of a vector space $V$, and $S$ satisfies the following conditions:

(i) $ax \in S$ whenever $x \in S$ for any scalar $a$

(ii) $x + y \in S$ whenever $x \in S$ and $y \in S$

then $S$ is said to be a **subspace** of $V$.

Condition (i) says that $S$ is closed under scalar multiplication. That is, whenever an element of $S$ is multiplied by a scalar, the result is an element of $S$. Condition (ii) says that $S$ is closed under addition. That is, the sum of two elements of $S$ is always an element of $S$. Thus, if we do arithmetic using the operations from $V$ and the elements of $S$, we will always end up with elements of $S$. A subspace of $V$, then, is a subset $S$ that is closed under the operations of $V$.

Let $S$ be a subspace of a vector space $V$. Using the operations of addition and scalar multiplication as defined on $V$, we can form a new mathematical system with $S$ as the universal set. It is easily seen that all eight axioms will remain valid for this new system. Axioms A3 and A4 follow from Theorem 3.1.1 and condition (i) of the definition of a subspace. The remaining six axioms are valid for any elements of $V$, so in particular they are valid for the elements of $S$. Thus every subspace is a vector space in its own right.

**Remark.** In a vector space $V$ it can be readily verified that $\{0\}$ and $V$ are subspaces of $V$. All other subspaces are referred to as **proper subspaces**.