Due Monday April 4th: write a short summary of one of the possible projects and post on web in Discussion Board (cut and paste).

Due Wednesday April 6th:

I. Vector Spaces: Check that all eight axioms of a vector space (see handout from class) hold for the set \( C[R] = \) set of continuous functions defined on the real numbers. Show what the various axioms mean when applied to the function \( x(t) = \sin t \), \( y(t) = t^2 \), \( z(t) = e^t \). Use the scalars \( \alpha = 2 \), \( \beta = 7 \). The goal is to get used to interpreting the vector space axioms when the elements of the space are functions.

II. Complex Numbers and Euler’s Formula: \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \). Where does this formula come from? Taylor series!

Recall that the Taylor series for a function \( f(x) \) based at the point \( x_0 \) is given by

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \ldots + \frac{1}{n!}f^n(x_0)(x-x_0)^n + \ldots
\]

Using this formula, one gets the following:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots
\]

C1. For \( f(x) = e^x \) and \( x_0 = 0 \), calculate the Taylor series. Show that you get the formula given above.

C2. Get the Taylor series for \( e^{i\theta} \) by substituting \( i\theta \) for \( x \) in the formula for \( e^x \). Simplify by using that \( i^2 = -1 \). Show that the series for \( e^{i\theta} \) is equal to the series for \( \cos(\theta) \) plus \( i \) times the series for \( \sin(\theta) \). In other words, show that \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \) by using the Taylor series formulas.

C3. Rewrite \( e^{4+3i} \) as \( z = u+iv = \text{Re } z + i \text{ Im } z \) using Euler’s formula.

C4. a. Use the guess and check method to solve the equation \( x''(t) + x'(t) + x(t) = 0 \).

b. You should get solutions of the form \( x_1(t) = e^{(a+ib)t} \) and \( x_2(t) = e^{(a-ib)t} \). To see what it means to have complex exponential solutions, use Euler’s formula to rewrite \( x_1(t) \) in the form \( x_1(t) = u(t) + iv(t) \).

What do \( u(t) \) and \( v(t) \) equal?
We say that \( u(t) \) is the real part of the complex number \( x_1(t) \) and \( v(t) \) is the imaginary part of the complex number \( v(t) \). We write this as \( x_1(t) = Re(x_1(t)) + i Im(x_1(t)) \).

c. Show that the function \( u(t) \) solves the DE \( x''(t) + x'(t) + x(t) = 0 \).

d. Show that the function \( v(t) \) solves the DE \( x''(t) + x'(t) + x(t) = 0 \).

Conclusion: If \( Y(t) = e^{\lambda t} \vec{v} \) solves the DE with \( \lambda \) complex, then \( Y_1(t) = Re(Y(t)) \) and \( Y_2(t) = Im(Y(t)) \) are two independent real solutions of the DE.

III. Linearity: A function \( f(x) \) (or a transformation \( T \) or a matrix \( M \) or an ‘operator’ \( D \)) is **linear** if it satisfies the following properties:

i. \( f(c_1 x) = c_1 f(x) \), where \( c_1 \) is any scalar (i.e. constant number).

ii. \( f(x_1 + x_2) = f(x_1) + f(x_2) \).

iii. \( f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2) \), where \( c_1, c_2 \) are scalars (i.e. constant number).

L1. Show that if condition (i) and (ii) hold then condition (iii) must hold.

L2. Show that if condition (iii) holds then conditions (i) and (ii) must hold.

L1 and L2 together prove the statement: “(i) and (ii) are true if and only if (iii) is true.”

Claim: Matrices are linear objects.

Let \( M = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \). Let \( \vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \). Let \( \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \).

L3. Calculate \( M(7\vec{v}_1) \) and \( 7M(\vec{v}_1) \) and show that they are equal.

L4. Calculate \( M(\vec{v}_1 + \vec{v}_2) \) and \( M(\vec{v}_1) + M(\vec{v}_2) \) and show that they are equal.

These calculations suggest that a matrix satisfies properties

(i) \( M(c_1 \vec{v}_1) = c_1 M(\vec{v}_1) \)

(ii) \( M(\vec{v}_1 + \vec{v}_2) = M(\vec{v}_1) + M(\vec{v}_2) \)

Optional Challenge Problem: Prove (i) and (ii) for the general case

Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( a, b, c, d \) are any real numbers. Let \( \vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \). Let \( \vec{w} = \begin{pmatrix} w_x \\ w_y \end{pmatrix} \).
Let $c_1$ be any real number.

L5. Calculate $M(c_1 \vec{v})$ and $c_1 M(\vec{v})$ and show that they are equal.

L6. Calculate $M(\vec{v} + \vec{w})$ and $M(\vec{v}) + M(\vec{w})$ and show that they are equal.

These calculations show that a $2 \times 2$ matrix satisfies properties (i) and (ii) and hence is linear.

4. Consider the third order linear system

$$x'''(t) + 4x''(t) - 2x'(t) + 7x(t) = 0.$$

a. Rewrite this as a system of three first order equations in terms of the variables $(x(t), v(t), a(t))$, where $v(t)$ is velocity and $a(t)$ is acceleration.

b. Write this system in matrix form.

5. For $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$, we found that the general solution to the linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$ was

$$\vec{Y}_g(t) = c_1 \vec{Y}_1(t) + c_2 \vec{Y}_2(t) = c_1 e^{2t} \vec{v}_1 + c_2 e^{-2t} \vec{v}_2$$

where $\vec{v}_1 = (3, 1)$ and $\vec{v}_2 = (-1, 1)$.

a. Determine the particular solutions with initial conditions a) $\vec{Y}(0) = (3, 1)$, (b) $\vec{Y}(0) = (-3, -1)$. (c) $\vec{Y}(0) = (-1, 1)$, (d) $\vec{Y}(0) = (1, -1)$. These are "easy"; just consider the initial conditions for the solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$.

Graph these solutions on the phase line diagram. Use graph paper so your curves are accurate.

b. In class, we found that for the particular solution with initial conditions $\vec{Y}'(0) = (2, 1)$ the value of the constants are $c_1 = 3/4$ and $c_2 = 1/4$.

Find the solutions that satisfy each of the following initial conditions: (a) $\vec{Y}(0) = (-2, 1)$, (b)$\vec{Y}(0) = (-1, -2)$. (c) $\vec{Y}(0) = (2, -1)$.

Add these curves to the phase portrait diagram that you started in (a). Draw these four initial points and then draw the solutions curves that go through them.

Sect 3.1 #5, 7, 8,

Challenge problem: #14.

Sect 3.2 For these you can use Mathematica to calculate eigenvectors and eigenvalues. Do a double
check by hand that you have paired up the eigenvector with its correct eigenvalue.

#2, 7, 14 (first determine the general solution). For each of these problems, classify the type of the equilibrium point at the origin. Do these examples agree with our conjectures that related the eigenvalues to the type of equilibrium?