Concavity Test: if $f''(x) > 0$ then graph of $f$ is concave up.
if $f''(x) < 0$ then graph of $f$ is concave down.

**Example:**

Determine concavity:

1) $f'(x) = -x^4 + 2x^2 + 2$

To find $f''(x)$:

$f''(x) = -4x^3 + 4x = 4x(x^2 - 1) = 0$

- $x = 0$
- $x^2 - 1 = 0$

**Critical Points:**

- $x = -1$
- $x = 1$
- $x = 0$

**Sign Chart:**

<table>
<thead>
<tr>
<th>Interval</th>
<th>$f''(x)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>-</td>
<td>concave down</td>
</tr>
<tr>
<td>$-1 &lt; x &lt; 1$</td>
<td>+</td>
<td>local min</td>
</tr>
<tr>
<td>$x &gt; 1$</td>
<td>+</td>
<td>concave up</td>
</tr>
</tbody>
</table>

**Derivative Test:**

- Local min at $x = 0$

2) $f''(x) = -12x^2 + 4 = 4(-3x^2 + 1) = 0$

- $-3x^2 + 1 = 0$
- $3x^2 = 1$
- $x^2 = \frac{1}{3}$
- $x = \pm \sqrt{\frac{1}{3}}$

**Critical Points:**

- $x = -\sqrt{\frac{1}{3}}$
- $x = 0$
- $x = \sqrt{\frac{1}{3}}$

**Sign Chart:**

<table>
<thead>
<tr>
<th>Interval</th>
<th>$f''(x)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -\sqrt{\frac{1}{3}}$</td>
<td>-</td>
<td>concave down</td>
</tr>
<tr>
<td>$-\sqrt{\frac{1}{3}} &lt; x &lt; 0$</td>
<td>+</td>
<td>local min</td>
</tr>
<tr>
<td>$x &gt; 0$</td>
<td>+</td>
<td>concave up</td>
</tr>
</tbody>
</table>

**Derivative Test:**

- Local min at $x = 0$

*Inference Points:*

<table>
<thead>
<tr>
<th>Point</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\sqrt{\frac{1}{3}}$</td>
<td>$f(-\sqrt{\frac{1}{3}})$</td>
</tr>
<tr>
<td>$0$</td>
<td>$f(0)$</td>
</tr>
<tr>
<td>$\sqrt{\frac{1}{3}}$</td>
<td>$f(\sqrt{\frac{1}{3}})$</td>
</tr>
</tbody>
</table>
THE FIRST D

(a) If \( f'(x) \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
(b) If \( f'(x) \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
(c) If \( f'(x) \) does not change sign at \( c \) (for example, if \( f'(x) \) is positive on both sides of \( c \) or negative on both sides), then \( f \) has no local maximum or minimum at \( c \).

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of \( f'(x) \) changes from positive to negative at \( c \), \( f \) is increasing to the left of \( c \) and decreasing to the right of \( c \). It follows that \( f \) has a local maximum at \( c \).

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.

**EXAMPLE 2** Find the local minimum and maximum values of the function \( f \) in Example 1.

**SOLUTION** From the chart in the solution to Example 1 we see that \( f'(x) \) changes from negative to positive at \(-1 \), so \( f(-1) = 0 \) is a local minimum value by the First Derivative Test. Similarly, \( f'(x) \) changes from negative to positive at \( 2 \), so \( f(2) = -27 \) is also a local minimum value. As previously noted, \( f(0) = 5 \) is a local maximum value because \( f'(x) \) changes from positive to negative at \( 0 \).

**EXAMPLE 3** Find the local maximum and minimum values of the function

\[
g(x) = x + 2 \sin x \quad 0 \leq x \leq 2\pi
\]

**SOLUTION** To find the critical numbers of \( g \), we differentiate:

\[
g'(x) = 1 + 2 \cos x
\]

So \( g'(x) = 0 \) when \( \cos x = -\frac{1}{2} \). The solutions of this equation are \( 2\pi/3 \) and \( 4\pi/3 \). Because \( g \) is differentiable everywhere, the only critical numbers are \( 2\pi/3 \) and \( 4\pi/3 \) and so we analyze \( g \) in the following table.

<table>
<thead>
<tr>
<th>Interval</th>
<th>( g'(x) = 1 + 2 \cos x )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; x &lt; 2\pi/3 )</td>
<td>( + )</td>
<td>increasing on ( (0, 2\pi/3) )</td>
</tr>
<tr>
<td>( 2\pi/3 &lt; x &lt; 4\pi/3 )</td>
<td>( - )</td>
<td>decreasing on ( (2\pi/3, 4\pi/3) )</td>
</tr>
<tr>
<td>( 4\pi/3 &lt; x &lt; 2\pi )</td>
<td>( + )</td>
<td>increasing on ( (4\pi/3, 2\pi) )</td>
</tr>
</tbody>
</table>
\[\begin{align*}
\text{In}[1]:= & \quad f[x_] := -x^4 + 2x^2 + 2 \\
\text{In}[2]:= & \quad f[-1] \\
\text{Out}[2]= & \quad 3 \\
\text{In}[3]:= & \quad f[-1/\sqrt{3}] \\
\text{Out}[3]= & \quad \frac{23}{9} \\
\text{In}[4]:= & \quad f[+1/\sqrt{3}] \\
\text{Out}[4]= & \quad \frac{23}{9} \\
\text{In}[6]:= & \quad \text{N}[-1/\sqrt{3}] \\
\text{Out}[6]= & \quad -0.57735 \\
\text{In}[9]:= & \quad \text{Solve}\[f[x] = 0, \: x\] \\
\text{Out}[9]= & \quad \{\{x \to -i \sqrt{1 - 1 + \sqrt{3}}\}, \{x \to i \sqrt{1 + 1 + \sqrt{3}}\}, \{x \to -\sqrt{1 + 1 + \sqrt{3}}\}, \{x \to \sqrt{1 + 1 + \sqrt{3}}\}\} \\
\text{In}[10]:= & \quad \text{NSolve}\[f[x] = 0, \: x\] \\
\text{Out}[10]= & \quad \{\{x \to -1.65289\}, \{x \to 0. - 0.8556 \text{ i}\}, \{x \to 0. + 0.8556 \text{ i}\}, \{x \to 1.65289\}\}\end{align*}\]