1. \( s_n = 1 - \frac{1}{n} \)  
   \( s_1 = 0, s_2 = \frac{1}{2}, s_3 = \frac{2}{3} \)  
   An increasing sequence, bounded by 1.  
   Non-trivial convergence. Theorem says it will converge.  
   \( \lim_{n \to \infty} s_n = 1. \)

2. \( s_n = 1 + \frac{1}{n} \)  
   \( s_1 = 2, s_2 = \frac{3}{2}, s_3 = \frac{4}{3} \)  
   Decreasing sequence, bounded below by 0  
   and \( 15 = 2 = 2 \).  
   Non-trivial convergence. Theorem says it will converge.

3. \( s_n = n \)  
   \( s_1 = 1, s_2 = 2, s_3 = 3 \)  
   Sequences is unbounded.  
   The sequence diverges.
\[ \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 4, 1\frac{1}{2}, 2\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \ldots \]

a) upper bound \( U = 5 / 4, 2\frac{1}{2} \)
   least upper bound = \( 2\frac{1}{2} \)

b) lower bound \( L = -3, 0, -2 \)
   greatest lower bound = \( 0 \)
S3. If sequence is Cauchy, then it is bounded.

(i) Assume all $s_n \neq 0$.

**Def:** Let $\varepsilon > 0$. Then there exist $N$

such that for all $m, n \geq N$

$$|s_m - s_n| < \varepsilon$$

Take $m = N + 1$. Then $|s_m - s_N| < \varepsilon$

$$\Rightarrow s_m < 1 + s_N$$

if $m > N$,

let $M = \max \{ s_1, s_2, \ldots, s_N, s_{N+1} + \varepsilon \}$

Then $0 \leq s_m \leq M$ for all $m$.

(ii) General case: $M = \max \{ |s_1|, |s_2|, \ldots, |s_N|, |s_{N+1}| + \varepsilon \}$

$$\Rightarrow |s_m| < M$$ for all $m$.  

5.4. \( Q \) is not complete. Since there exist Cauchy sequences \( s_n \in Q \) s.t.
\[
\lim_{n \to \infty} s_n \text{ does not exist (in } Q \).
\]

e.g. \( s_1 = 3.1 \)
\[
\begin{align*}
s_2 &= 3.14 \\
s_3 &= 3.141 \\
\end{align*}
\]
\( s_n \) is the first \( n \) decimal places of \( \pi \).

Then thought of as a sequence in \( \mathbb{R} \)
\[
\lim_{n \to \infty} s_n = \pi \text{ and the limit is unique.}
\]
Since a convergent sequence is Cauchy, this sequence is Cauchy.

But \( s_n \) \( \not\to \) \( \pi \) \( \in \) \( \mathbb{Q} \). The sequence doesn't converge in \( \mathbb{Q} \).

\[
\begin{align*}
1 &+ 0 \\
\end{align*}
\]
\[
\begin{align*}
s, s_2, s_3, s_4 \in \mathbb{Q} \\
\end{align*}
\]

\( X = (1, 4) \). Then the sequence
\[
\begin{align*}
s_n &= 4 - \frac{1}{n} \text{ is Cauchy} \\
\end{align*}
\]

but doesn't converge to an element of \( X \).

So \( X \) is not complete.
5. Claim: If the least upper bound of sets exists then it is unique.

Proof: Let $u_1$ and $u_2$ both be upper bounds.

By condition (ii) of lub, if $u$ is any upper bound of $S$ then

$u_1, u_2 \leq u$, since $u_1, u_2$ are upper bounds

But $u_2$ is an upper bound of $S$ if

(i) $u_1 \leq u_2$.

Now consider $u_2$ as lub $S$. Then

$u_2 \leq u$ for any upper bound of $S$.

(ii) $u_2 \leq u$, since $u_2$ is upper bound of $S$.

(iii) $r(x) \rightarrow u_1 = u_2$
Claim: if \( u \) is lub \( S \) then for every \( \varepsilon > 0 \) there exist \( s \in S \) s.t. \( s \in [u - \varepsilon, u] \).

\[
\begin{align*}
\text{P.F.} \quad & \text{Assume* That there is not The case,} \quad \\
& \text{(by contradiction) i.e. That for some } \varepsilon > 0 \text{, there is no} \\
& \quad s \in [u - \varepsilon, u]. \\
\frac{u - s}{u - s} \quad & \text{Sinee } u \text{ is an upper bound for } S \\
& u > s \text{ for all } s \in S. \\
& \text{If no } s \text{ satisfies } u - \varepsilon < s \leq u, \\
& \quad \text{Then } u - \varepsilon \text{ is an upper bound for } S \\
& \quad s \leq u - \varepsilon \text{ for all } s \in S. \\
& \text{But Then we have an upper bound } (u - \varepsilon) \\
& \quad \text{which satisfies} \\
& \quad (u - \varepsilon) < u. \\
& \text{This contradicts That } u \text{ is the least} \\
& \quad \text{upper bound.} \\
& \text{Hence The assumption* is false.} \\
\end{align*}
\]

So for every \( \varepsilon > 0 \), \( \exists s \in S \) at \\
\( u - \varepsilon \leq s \leq u. \)
for all \( n \), \( u_n \) is an upper bound for \( S \).

Claim: if \( u \) is an upper bound for \( S \), then \( u \) is an upper bound for \( S \).

**Proof:** Assume \( u \) is not an upper bound for \( S \). Then there exists \( s \in S \) such that \( u < s < u_n \) for all \( n \).

\[
\frac{s - u}{2} \quad \frac{1}{u_n} \quad \text{will get a contradiction with}\ u_n \rightarrow u.
\]

Choose \( \varepsilon = \frac{s - u}{2} \). Then \( u_n \notin (u - \varepsilon, u + \varepsilon) \)

for all \( n \). Now assume \( u + \varepsilon = u + \frac{(s - u)}{2} = u + \frac{s - u}{2} = \frac{u + s}{2} > \frac{s}{2} = s \)

so \( u + \varepsilon < s < u_n \).

But if \( u_n \notin (u - \varepsilon, u + \varepsilon) \) for this choice of \( \varepsilon \), then \( u_n \) \( \not\rightarrow u \). This is a contradiction.

So \( u \) is an upper bound for \( S \).
7. \( a_n = (-1)^n \) 
\(-1, 1, -1, 1, -1, 1, \ldots\)

Subsequence: \( a_{2k} \) where \( n_k = 2k \) 
\( S_2, S_4, S_6, S_8, \ldots \)
\( 1, 1, 1, 1, \ldots \)

\( \lim \sup a_n = 1 \)

Subsequence: 
\(-1, -1, -1, -1, \ldots\)
\( S_2^\prime, S_5, S_8, S_{11}, S_{14}, \ldots, S_{2k}, \ldots, n_k = 2k - 1 \)

\( \lim \inf a_n = -1 \)

\( \lim \sup a_n = 1 \)

8. \( 1, -1, 1, -1, 2, -2, 3, -3, \ldots \)

Subsequence: \( 1, 2, 3, 4, \ldots \) \( n_k = 2k - 1 \), \( S_{2k} \).

\( \lim \inf S_k = +\infty \) (i.e., There is a subsequence) That diverges to \( +\infty \).
c. \( a_n = n^2 \)
-1, -4, -9, -16, ...

Every subseq. diverges to \(-\infty\).
So \( \lim \sup a_n = -\infty \).

d. \( a_n = \sin \left( \frac{n\pi}{6} \right) \)
\( \sin \left( \frac{\pi}{6} \right), \sin \left( \frac{\pi}{6} \right), \sin \left( \frac{\pi}{6} \right), \sin \left( \frac{3\pi}{6} \right), \sin \left( \frac{\pi}{6} \right) \)
\( \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \)

\( \sin \left( \frac{\pi}{2} \right) = 1/2 \)
\( \sin \left( \frac{\pi}{3} \right) = \sqrt{3}/2 \)

Sequence: \( \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1 \)

\( \sim 0 \), Then repeats

Closest subsequence: \( 1, 1, 1, 1, ... \rightarrow 1 \).
\( a_{n_k} \) with \( n_k = 3 + 12(k-1) \)

\( \lim \sup a_n = 1 \).
\[ e_{\frac{1}{n}}: \frac{1}{2}, \frac{3}{4}, 1, \frac{7}{4}, 2, \frac{15}{8}, 2\frac{1}{4}, \ldots \]

Choose subsequence

\[ 2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{7}{8}, \ldots, 2\frac{1}{2^k} \rightarrow 2 \]

So \( \limsup s_n = 2 \)

Subsequence \( s_{n_k} \), with \( n_k = 3^k \).