S.1. a. Give an example of a bounded increasing sequence. What does the Monotone Convergence Theorem say about this sequence? Can you determine the limit of this sequence?

b. Give an example of a bounded decreasing sequence. What does the Monotone Convergence Theorem say about this sequence? Can you determine the limit of this sequence?

c. Give an example of an unbounded increasing sequence. What is true about the limit of this sequence?

S.2. Consider the sequence given in Morgan, Ch. 8, p. 39 #7e.

(a.) Give three different upper bounds for this sequence. What is the least upper bound of this sequence?

b. Give three different lower bounds for this sequence. What is the greatest lower bound of this sequence?

I. Cauchy Sequences.

S3. Prove that if a sequence is Cauchy, then the sequence is bounded. Hint: Review the proof that if a sequence has a limit, then the sequence is bounded. Similar approach.

S4. Complete Spaces: a. Explain why the set \( \mathbb{Q} \) is not complete. b. Make up your own example of a set \( S \subset \mathbb{R} \) that is not complete. Explain.

II. Sup = Least Upper Bound and Inf = Greatest Lower Bound.

S5. Prove that the least upper bound of a set, when it exists, is unique. Assume that \( u_1 \) and \( u_2 \) are both least upper bounds for \( S \) and prove that \( u_1 = u_2 \). To do this, you will need to use the two conditions in the definition of least upper bound.

S6. Assume that \( S \subset \mathbb{R} \) is a bounded set and that \( u \) is an upper bound for \( S \). Prove that if \( u \) is a least upper bound for \( S \) then, for every \( \epsilon > 0 \), there exists a point \( s \in S \) such that \( s \in [u-\epsilon,u] \). Hint: Try proof by contradiction. Assume that this not the case and get a contradiction.

S7. Assume that for all \( n \in \mathbb{N} \), the number \( u_n \) is a upper bound for a set \( S \). Assume that \( \lim_{n \to \infty} u_n \) exists and \( \lim_{n \to \infty} u_n = u \). Prove that \( u \) is an upper bound for \( S \).

III. Subsequences:

Morgan, Ch. 8, p. 39 #7(do all except f). Note: when we say the lim sup = \( +\infty \), we mean that there is a subsequence whose values diverge to \( +\infty \). For each of these examples, write out the subsequence that produces the lim sup. Use the notation \( s_{nk} \) for a subsequence and give the
formula $n_k$ which gives the index of the $k$th element of the subsequence in terms of the $n_k$ th element of the original sequence.