Math 301 Homework, Donnay, Fall 301: Assignment #Wk 5-6:

Part 1:

S3. c. We can write a natural number \( n \) in base 3 as

\[
    n = b_n b_{n-1} \ldots b_2 b_1 b_0 = b_n(3^n) + b_{n-1}(3^{n-1}) + \cdots + b_2(3^2) + b_1(3^1) + b_0(3^0).
\]

Write out the base ten number \( n = 152 \) in base three. Hint: Look at the Tom Lehrer’s video and song about base 8 linked on our play by play website.

Hint: Look at the different powers of 3: \( 3^6, 3^5, 3^4, 3^3, 3^2, 3^1, 3^0 \). How many times do these go into 152? What is the remainder?

Part 2:

S1. Prove (via mathematical induction) that for all positive \( n \in \mathbb{N} \),

\[
    1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.
\]

Hint: For the induction step, write out what the statement is when \( n = k+1 \). Go from both sides towards the middle. Recall the formula we proved in class for \( \sum_{i=1}^n i \) which gives the sum of the first \( n \) integers.

S2. Prove that the set of all accumulation points of the irrational numbers is all of \( \mathbb{R} \).

Hint: We can not use the result that says that in every interval there is a irrational number. That result follows from this statement and the \( \epsilon \)- theorem we have about accumulation points. Let \( x \in \mathbb{R} \) be either rational or irrational. The definition of accumulation point says we need a sequence of irrational numbers \( s_n \) that converges to \( x \) and the \( s_n \) are all irrational and \( s_n \neq x \).

There are several approaches you could use:

Approach 1: Recall the suggestion Jules made in class: make a sequence with the \( \frac{1}{n} \) technique: \( s_n = x + \frac{1}{n} \). However if \( x \) is rational, then \( s_n \) would also be rational and we need \( s_n \) to be irrational. So we could try something like \( s_n = x + \frac{\sqrt{2}}{n} \). In writing out your proof, you will need to justify why the \( s_n \) are irrational. (Assume they are rational and get a contradiction).

Approach 2: Write out \( x \) in decimal notation: \( x = k.d_1d_2d_3\ldots d_n\ldots \) where \( k \) is the integer part of \( x \). Then create a sequence \( s_n \) of irrational numbers. The \( s_n \) equals \( x \) up to the first \( n \) decimal places but afterwards continues as the decimal expansion of some irrational number. For example, use \( \pi = 3.1415\ldots = 3.p_1p_2p_3\ldots p_n\ldots \).

S3. Prove that: If for every \( \epsilon > 0 \), the interval \( (p - \epsilon, p + \epsilon) \) contains a point of \( S \setminus \{p\} \) then \( p \) is an accumulation point of the set \( S \).

Hint: Let \( \epsilon = \frac{1}{n} \). For this value of \( \epsilon \), use the above assumption to get a point \( s_n \). Do this for every \( n \in \mathbb{N} \). Prove that the resulting sequence converges to \( p \).
Morgan Ch. 3, #6 Using the definition of convergence, prove that this sequence does not converge to 0. You will need to figure out what the negation of the definition of a limit of sequence is: i.e. show that \( \lim_{n \to \infty} s_n \neq 0 \). Extra credit: Prove that for any number \( L \in \mathbb{R} \), the sequence does not converge to \( L \). This shows that the sequence does not converge.