Midterm exam will take place during the week of Oct. 3 - 7th. I will announce the exact details (in class or take home, day) next week. The exam will cover all the material we have done so far: iteration, dynamical systems, countable/uncountable (Ch. 2), sequences and limits (Ch. 3). The exam will include material on this present hw assignment.

Part 0: Test Review Assignment: to be done for Monday/Tuesday class and brought to class. You will meet in your groups and share your review document with your teammates to see how your approach compares to theirs. Prepare a review document about countable/uncountable that has the following sections:

1. Definitions (ex. a set is countable if .... A set B has the same cardinality as A if ...).
2. Main Theorems (State them. Ex. The union of countably many countable sets is countable, etc.)
3. Examples (give simple examples that illustrate the main concepts; ex. countable set, ex. uncountable set, etc.).
4. List some types of problems that might be given on the exam.

Part 1: Due Wednesday Sept 28th by 5pm.

S1. in class we proved: Theorem: If a sequence $s_n$ converges then the sequence is bounded.
   a. State the converse of this theorem. Is the converse true (give a proof) or false (give a counter-example).
   b. State the contraposition of this theorem. Is the contraposition true (explain why) or false (give a counter-example)
   c. Using the contraposition to justify your result, give an example of a sequence that diverges.

S2. Prove rigorously that $\lim_{n \to \infty} \frac{1}{n} = 0$.

Morgan Ch. 3 #7 Use the limit laws to justify every step you take in determining the limit. You are only allowed to use the following fact which you have just proven rigorously: $\lim_{n \to \infty} \frac{1}{n} = 0$.

#8 Justify your answer in terms of growth rates but do not give an $\varepsilon$ proof.

Part 2: Due Friday Sept 30 by 5pm.

S1. Prove (via $\varepsilon$ and $N$) that the constant sequence $s_n = 2$ for all $n \in N$ converges to 2. Hint: Draw your target zone picture to determine the waiting time.

S2. Determine the limit of $s_n = 4 - 7e^{-n}$. Prove this limit using the rigorous definition (not using the limit laws).

Morgan Ch. 3 #9 Justify your answer in terms of growth rates but do not give an $\varepsilon$ proof.
#13, 15.
5.1: Theorem: If sequence $s_n$ converges then it is bounded

a. Converse: If a sequence $s_n$ is bounded then it converges.

   False. Let $s_n = (-1)^n \cdot (-1, 1, -1, 1, -1, 1, \ldots)$ which is bounded sequence but does not converge.

b. Contraposition. If seq $s_n$ is not bounded then it does not converge.

or i.e. If seq. $s_n$ is unbounded then it diverges.

   True since the contraposition is equivalent to the original statement which is true.

c. Let $s_n = n$. Then the sequence is unbounded so it diverges.
Ch 3.47. \lim_{n \to \infty} \frac{2n^2 + 5n + 1}{7n^2 + 4n + 3} = \lim_{n \to \infty} \frac{2 + \frac{5}{n} + \frac{1}{n^2}}{7 + \frac{4}{n} + \frac{3}{n^2}} \text{ by n}^2 \\
= \frac{\lim_{n \to \infty} \left(2 + \frac{5}{n} + \frac{1}{n^2}\right)}{\lim_{n \to \infty} \left(7 + \frac{4}{n} + \frac{3}{n^2}\right)} \text{ by (1.2.4), assuming limit exists} \\
= \frac{\lim_{n \to \infty} 2 + \lim_{n \to \infty} \frac{5}{n} - \lim_{n \to \infty} \frac{1}{n^2}}{\lim_{n \to \infty} 7 + \lim_{n \to \infty} \frac{4}{n} + \lim_{n \to \infty} \frac{3}{n^2}} \text{ by (1.2.4). assuming limits exist} \\
= \frac{2 + 5 \lim_{n \to \infty} \frac{1}{n} + \left(\lim_{n \to \infty} \frac{1}{n}\right)\left(\lim_{n \to \infty} \frac{1}{n}\right)}{7 + 4 \lim_{n \to \infty} \frac{1}{n} + 3 \left(\lim_{n \to \infty} \frac{1}{n}\right)\left(\lim_{n \to \infty} \frac{1}{n}\right)} \text{ by limit of constant (1.2.1), (1.2.3) assuming all limits exist} \\
= \frac{2 + 5 \cdot 0 + 0.0}{7 + 4 \cdot 0 + 3 \cdot 0.0} \text{ since } \lim_{n \to \infty} \frac{1}{n} = 0 \\
= \frac{2}{7} \text{ all the limits exist} \\
\text{Intuition: leading growth terms of } 2n^2 + 5n + 1 \text{ and } 7n^2 = \frac{2n^2}{7n^2 + 4n + 3} \text{ and } 7n^2
\[ a_n = \frac{e^n}{n^5 + n - 5} = \frac{e^n}{n^5 \left(1 + \frac{1}{n^4} - \frac{5}{n^2}\right)} \]

Then

\[ \frac{e^n}{n^5} \]

\[ e^n > n^5 \]

so \( \frac{e^n}{n^5} \to \infty \)

\[ \text{Hence the seq diverges.} \]
S1. Prove that the constant sequence \( s_n = 2 \) for \( n \geq 1 \) converges to \( L = 2 \).

Let \( \varepsilon > 0 \). Let \( N = 0 \) (since \( N \geq 1 \)). Then if \( n > N \),
\[
|s_n - 2| = |2 - 2| = 0 < \varepsilon.
\]

So \( s_n \to 2 \).
\[ S_n = 4 - 7e^{-n} = 4 - \frac{7}{e^n}, \quad n \to \infty \quad \text{since} \quad \frac{7}{e^n} \to 0 \]

**Preliminary Calculation:**

\[ |S_n - 1| = |4 - \frac{7}{e^n} - 4| = \left| -\frac{7}{e^n} \right| = \frac{7}{e^n} < \varepsilon \]

\[ \iff \frac{e^n}{\varepsilon} > \frac{1}{\varepsilon} \]

\[ \Rightarrow e^n > \frac{1}{\varepsilon} \]

\[ \Rightarrow \ln(e^n) > \ln(1/\varepsilon) \]

\[ \Rightarrow n > \ln(1/\varepsilon). \]

Let \( \varepsilon > 0 \). Let \( N = \ln(1/\varepsilon) \). If \( n > N \)

Then \( |S_n - 1| = |4 - \frac{7}{e^n} - 4| < \varepsilon \)

by the above calculation.
Exercise 3.9: \( a_n = \frac{2^n}{n!} \)

By rate of growth, \( 2^n \ll n! \)

so \( \frac{2^n}{n!} \rightarrow 0 \).
13. Prove: If \( a_n \to a \) and \( b_n \to b \) then \( a_n + b_n \to a + b \).

\[
\text{Scratch: will ultimately want} \\
\left| (a_n + b_n) - (a+b) \right| < \varepsilon
\]

\[
\left| (a_n + b_n) - (a+b) \right| < \left| a_n - a \right| + \left| b_n - b \right|
\]

\[
\begin{align*}
\text{Define} & \quad \varepsilon_1 \\
\text{Make these} & \quad < \varepsilon_2
\end{align*}
\]

**Proof** Fix an arbitrary \( \varepsilon > 0 \).

**WTS:** \( \exists N \text{ s.t. } n > N \Rightarrow \left| (a_n + b_n) - (a+b) \right| < \varepsilon \).

We know \( a_n \to a \). So \( \forall \varepsilon > 0 \exists N_1 \text{ s.t. } n > N_1 \Rightarrow \left| a_n - a \right| < \varepsilon_1 \).

In particular for \( \varepsilon_1 = \varepsilon/2 \) (for \( \varepsilon \) fixed as above),

\[
\exists N_1 \text{ s.t. } n > N_1 \Rightarrow \left| a_n - a \right| < \varepsilon_1 = \varepsilon/2
\]

Similarly since \( b_n \to b \), \( \exists N_2 \text{ s.t. } n > N_2 \Rightarrow \left| b_n - b \right| < \varepsilon/2 \).

Now consider \( N = \text{max} \left\{ N_1, N_2 \right\} \). Then

\[
n > N \Rightarrow n > N_1 \text{ and } n > N_2 \Rightarrow \left| a_n - a \right| < \varepsilon_1 \text{ and } \left| b_n - b \right| < \varepsilon/2
\]

\[
\Rightarrow \left| (a_n + b_n) - (a+b) \right| = \left| a_n - a + b_n - b \right| \\
\leq \left| a_n - a \right| + \left| b_n - b \right| \\
< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \text{as desired!}
\]
Ch 3.15. Squeeze Theorem.

if \( a_n \leq b_n \leq c_n \) and \( a_n \to l \) and \( c_n \to l \)

Then \( \lim_{n \to \infty} b_n = l \) exists and \( b_n \to l \).

Goal: Show that given any \( \varepsilon > 0 \), there exists \( N \) s.t. if \( n > N \) then \( |b_n - l| < \varepsilon \).

Idea: \( a_n \to l \) so given \( \varepsilon > 0 \), there exists \( N_1 \) s.t. if \( n > N_1 \), then \( |a_n - l| < \varepsilon \).

\( c_n \to l \) so given \( \varepsilon > 0 \), there exists \( N_2 \) s.t. if \( n > N_2 \), then \( |c_n - l| < \varepsilon \).

Visual:

\[\begin{align*}
L + \varepsilon & \quad \cdots \quad c_n \quad \cdots \quad c_k \\
L & \quad \cdots \\
L - 3 & \quad \cdots \\
N_1 & \quad N_2 & \quad N & \quad n
\end{align*}\]

if \( n > \max\{N_1, N_2\} \) then \( a_n \in (L - \varepsilon, L + \varepsilon) \)

and \( c_n \in (L - \varepsilon, L + \varepsilon) \).

Since \( a_n \leq b_n \leq c_n \), we also get \( b_n \in (L - \varepsilon, L + \varepsilon) \).
Proof. Let $\varepsilon > 0$. Choose $N_1$ and $N_2$ st.

\[
\begin{cases}
|a_n - L| < \varepsilon & \text{if } n > N_1 \\
|c_n - L| < \varepsilon & \text{if } n > N_2.
\end{cases}
\]

Let $N = \max \{N_1, N_2\}$. Then if $n > N$, both

\[
|a_n - L| < \varepsilon \quad \text{and} \quad |c_n - L| < \varepsilon.
\]

ie

\[
L - \varepsilon < a_n < L + \varepsilon
\]

and

\[
L - \varepsilon < c_n < L + \varepsilon.
\]

So

\[
L - \varepsilon < a_n < b_n < c_n < L + \varepsilon
\]

\[
\begin{align*}
\frac{b_n - a_n}{c_n - a_n} & < \frac{L + \varepsilon - L - \varepsilon}{L + \varepsilon - L - \varepsilon} \\
& = 1.
\end{align*}
\]

So

\[
|b_n - L| < \varepsilon \quad \text{for all } n > N.
\]

So $b_n \to L$. 