Midterm 2 Review

The midterm will be a take home exam. It will consist of a two hour closed book portion and then an untimed open book portion. The closed book portion should be taken in a library like last time (unless you make special arrangements with me). So you can take the test at a Haverford library too.

The closed book test will be available starting next Wednesday. The open book portion will be given out in class on Wednesday/Thursday. Hand the entire test in at the start of class (either on Monday or Tuesday). Of course, no discussing the test with anyone one you start working on it.

For the closed book portion, you will be allowed to write notes on one side of a 3 x 5 index card. The notes must be hand written. You must bring the index card to class (after you have created it) and I will initial it. You must do this before you are allowed to use the index card on the closed book portion.

The first midterm covered material up through the end of Ch 3 but did not include accumulation points (p. 18).

For the 2nd Midterm, we will focus on the material since the first midterm although that material builds upon what was covered earlier. Thus the topics you should review include:

Proof by Induction

Cantor Set (Ch 13).

Accumulation Points of a set S (Ch. 3). Dense (ex. Q is dense in R and Irrationals are dense in R).

Negation of statements.

Limits of functions. (Ch. 4).

Open and Closed Sets (Ch. 5) and Boundary Points.

Continuous Functions (Ch. 6, Ch 7)
Overall Strategy:

For each concept, you should know the definition. (Ex. Open sets). You should be able to make calculations using the concept. You should know simple examples that illustrate the concept. These examples might involve pictures (visual representation) or formulas (analytical representation). You should prepare some questions that are of the type Prof Donnay might ask on the exam.

You should “know” the main theorems we have covered. Knowing has several components.

a. Be able to give the statement of the theorem.
b. Be able to give examples that illustrate what the theorem means.

c. Be able to apply the theorem to problems (similar to what you have done in the homework.)
d. Be able to give the proof of the theorem.

For the midterm, I will ask you to prove some of the theorems we have covered in class. Below is a list of the theorems that I might ask you to prove.

For your review for the test, you could start by going through your notes and the text book and making a list of what you think are the main concepts and associated theorems. This should be a brief list of just the main key items. Ex. Open and Closed Sets, Continuous functions would be on the list. Later you can go back in fill in a bunch of details associated with these concepts and theorems.

On the web, I will post my list. So after you have tried making your list, you can check it against my list. There is a certain amount of subjectivity in deciding what are the “key concepts and theorems” and what are secondary concepts. So do not worry about getting the choices exactly right: the important aspect is the process of thinking about what are the key ideas.
Key Concept:

Definition:

Examples (both visual and analytic):

Theorems:

Examples Illustrating Theorem (both when assumptions of theorem are satisfied and when they are not).

Possible ways the theorem can be used:
Key Concepts and Associated Theorems since Midterm 1:

Proof by Induction. Be able to do proof by induction.

Cantor Set (Ch 13): Geometric construction. Number of pieces at each level $C_n$. Width of these pieces. Connection with base three numbers to determine which regions are in $C_n$ and which points are in the final Cantor set C. Proof that Cantor set is uncountable.

Accumulation Points of a set $S$ (Ch. 3). Definition. Alternative characterization. $\mathbb{Q}$ is dense in $\mathbb{R}$ (and Irrationals are dense in $\mathbb{R}$).

Negation of statements.

Limits of functions. (Ch. 4). Definition. Be able to do a $\delta - \varepsilon$ proof for a limit.

Open and Closed Sets (Ch. 5) and Boundary Points. Definition of boundary point. Definition of open and closed sets in terms of boundary points. Alternative characterization of open sets. Combinations of open/closed sets.

Continuous Functions (Ch. 6).
- Limit definition. Then limit definition using $\delta - \varepsilon$.
- Theorems about combining continuous functions: composition, sum, product.
- Three equivalent definitions (6.1 Proposition)
Detailed list of Topics Covered:

Negation of statements; be able to give the negation of a statement.

Proof by Induction: be able to do these.

Cantor Set (Ch 13): Geometric construction. Number of pieces at each level $C_n$. Width of these pieces. Connection with base three numbers to determine which regions are in $C_n$ and which points are in the final Cantor set $C$. Proof that Cantor set is uncountable.

Ch. 3: Accumulation points (basic definition and alternative characterization). Definition in terms of limit point of sequences in $S$. Theorem: A point $p$ is a limit point of a set $S$ if and only if for each $\varepsilon > 0$ the interval $(p-\varepsilon, p+\varepsilon)$ contains a point of $S \setminus \{p\}$.

Ch 4: Functions and Limits: Know definition of limit of a function and be able to calculate the limit using $\delta - \varepsilon$. Both at a specific point $p=2$ and for a general point $p$. Be able to use the definition to prove that a limit does not exist. Be clear on the difference (and similarities) between the limit of a sequence and the limit of a function. Be able to make and understand examples with the weird functions: $\sin(1/x)$, $x \sin(1/x)$, $\chi_0(x) = \text{the characteristic function of the rationals}$ and more generally $\chi_S(x) = \text{the characteristic function of a set } S$. Know limit laws for functions. Be able to prove using $\delta - \varepsilon$ Prop 4.6.1 (constant law) and 4.6.2 (sum law).

Ch. 5: Topology = Open and Closed Sets. Know definitions of boundary point of a set, open set, closed set. Be able to use these definitions to determine what the boundary of a set equals and whether a set is open, closed, or neither (clopen). Ball $B(p, r) = \{w: \text{dist}(p, w) \leq r\}$. This is a closed ball. Know the alternative characterizations of open sets (the ball condition): A set $S$ is open iff about every point $p$ in $S$, there exists a ball that is completely contained in $S$. (Proposition 5.3). Know basic theorems about open and closed sets: a set is open iff and its complement is closed and vice versa. Know what happens when you combine (via union and intersection) open and closed sets in various ways. (Proposition 5.4) Be able to make up examples to illustrate the various possibilities. (Less important: interior of a set, closure of a set, isolated point).

Ch. 6: Continuous Functions:
Know the three equivalent definitions.
Given (a simple) function, be able to show that it is continuous using the $\delta - \varepsilon$ definition or using the sequence definition. (Ex. Show using the sequence definition that $f(x) = 3x^2 + 2x - 7$ is continuous using what we now know about sequences. Let $x_n \to p$, determine $\lim f(x_n)$).
Be able to use any of the three definitions to prove that a function is not continuous. Point of confusion: we say a function is continuous if it is continuous at every point in its domain. In the simplest cases, the domain is all of $\mathbb{R}$.

Ch. 7: Composition of Function:
Be able to calculate the composition of functions and determine the domain of a composition. Be able to prove continuity results about a composition of functions.
Proofs for Midterm 2:
You will be asked to do some proofs on the midterm. Some proofs will be in the closed book part; some proofs will be in the open book part. The possible proofs will include:

- Show that an arbitrary union and a finite intersection of open sets are open.
- Show that an arbitrary intersection and a finite union of closed sets are closed.
- Prove using the \( \delta - \varepsilon \) definition that
  \[
  \lim cf = c \lim f \quad \text{and} \quad \lim (f+g) = \lim f + \lim g.
  \]

Theorem: A point \( p \) is a limit point of a set \( S \) if and only if for each \( \varepsilon > 0 \) the interval \( (p-\varepsilon, p+\varepsilon) \) contains a point of \( S \setminus \{p\} \).

Theorem: A set \( S \) is open iff for every point \( p \) in \( S \), there exists a radius \( r \) such that the ball \( B(p, r) \) is contained in \( S \).

Composition of continuous fns if continuous
Continuity:
- If \( \delta - \varepsilon \) condition holds then sequence definition holds.
- The \( \delta - \varepsilon \) condition holds iff the open set condition holds.