Quiz due at 5 pm Monday. You can pick the quiz up from the basket outside my office. You will have 20 minutes to take the quiz. You must take the quiz in the library. The quiz is true/false and covers the basic material from the previous week’s classes.

Most of the HW set is due Friday Feb 20th (but do not wait till the last minute to start it!). There is a lot of Mathematica in this so questions 2, 3, 4 will all be team problems. Each team of 2 people will submit a single solution set. Do question 1 individually.

Due Wed Feb 18th.

1. In the Cantor Set construction:

Start with $C_0 = [0, 1]$. Remove the middle third to get the set $C_1 = [0, 1/3] \cup [2/3, 1]$. From each of these pieces, remove the middle third to get the set $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9]$. Continue in this manner. To form the set $C_n$, remove the middle third of each interval that made up $C_{n-1}$. The Cantor Set is defined as

$C = \cap_n C_n$.

So $C$ consists of those points that are in $C_n$ for all $n$.

a. Make a conjecture about the number of intervals that are in the set $C_n$. You may want to start by making a table in which you record $n$ and the number of intervals for $n=0, 1, 2, 3, 4$ to help your determine the pattern. Prove this conjecture using mathematical induction.

b. Make a conjecture about the length of each interval in the set $C_n$. Prove this conjecture using mathematical induction.

Due Friday Feb 20th

2. Error Bound. Consider the function $f(x) = \sin x$ and its Taylor polynomials $T_n(x)$ based at $x_0 = 0$. Only consider $n$ odd. Take the interval $x \in [-10, 10]$. Goal: determine $N$ such that $T_n(x)$ is a "good approximation" to $f(x)$ for all $x \in [-10, 10]$ and all $n \geq N$. By "good", let's say the error should be less than $\epsilon = .1$. Formally, we are asking to find $N$ such $|f(x) - T_n(x)| < \epsilon = .1$ for all $n \geq N$ and for all $x \in [-10, 10]$. (Note: here we are using $n \geq N$).

a. Use the Taylor Reminder formula to find a value for $N$. Note that this will probably not be the smallest possible value of $N$ since we make estimates in simplifying $R_n(x)$. That is ok. Our goal is to derive a result that is shown to be absolutely (rigorously) correct.

It might be helpful to use the Mathematica Table command: The syntax is shown by:

Table[n^2, {n, 1, 10}] or Table[{n, 2n+3}, {n, 1, 10}]

Factorial is simply 5! in Mathematica.

b. Use Mathematica to determine for which $n$, $|f(x) - T_n(x)| < \epsilon = .1$ for all $x \in [-10, 10]$ FAILS. Do this
by graphing \( f(x) \) and its \( \epsilon \) target zone and various values of \( T_n(x) \).

Start by first plotting the following to make sure you can get the target zone drawn.

\[
\text{Plot}\left[\{\sin[x], \sin[x]+.1, \sin[x]-.1\}, \{x, -10, 10\}\right]
\]

c. Also with Mathematica, determine visually the smallest \( n \) such that \( T_n(x) \) does appear to be in the \( \epsilon \) target zone. Note that a computer picture while it is evidence towards the result is not a rigorous proof. How does this smallest \( n \) value you got via computer evidence compare to the \( N \) value you got via your rigorous proof?

d. What is the value of doing the rigorous estimate vs checking visually?

3. For \( f(x) = e^x \) and \( x_0 = 0 \), consider the Taylor polynomials \( T_n(x) \).

a. What is the formula for \( R_n(x) \)?

b. Prove that on any interval \( x \in [-R, R] \), the remainder \( R_n(x) \to 0 \) uniformly. This proves that on \([-R, R]\), \( T_n(x) \to f(x) \) uniformly.

c. What effect does changing the size of the interval (i.e. changing \( R \)) have on your proof of uniform convergence?

d. Does \( R_n(x) \to 0 \) uniformly for all \( x \in (-\infty, \infty) \)? Think of drawing an \( \epsilon \) target zone around \( e^x \) for all \( x \in (-\infty, \infty) \). Can the polynomial \( T_n(x) \) lie in the target zone for all \( x \)?

4. Let \( f(x) = \frac{1}{x} \) and take \( x_0 = 1 \).

a. Determine \( f^k(x) \) for \( k = 0, 1, 2, 3, 4 \).

b. Make a conjecture about the formula for the nth derivative \( f^n(x) \). Prove this formula using mathematical induction.

c. Give the Taylor Polynomials \( T_1, T_2, T_3, T_4 \) and then give the formula for \( T_n(x) \).

You can check your work using Mathematica. The following command calculates the Taylor Series for \( f(x) \) based at \( x_0 \) up to the nth term which is effectively \( T_n(x) \).

General syntax: \( \text{Series}[f(x), \{x, x_0, n\}] \)

Ex. \( \text{Series}[1/x, \{x, 1, 5\}] \)

d. Predict on what interval the Taylor polynomials will converge to the function: i.e. for what \( x \) does \( T_n(x) \to f(x) \). As part of your investigation of this questions, graph \( T_1, T_2, T_3, T_4 \) and \( f(x) \) on the same graphs via Mathematica.

e. (Challenge) See if you can prove some type of rigorous result using the Taylor Remainder Theorem.