STAIRCASE ITERATION

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Dynamical system generated via iteration of a function \( y = f(x) \). We start with an initial value \( x_0 \), then use the function \( f \) to get the successive values in the sequence:

\[
x_1 = f(x_0), \ x_2 = f(x_1), \ x_3 = f(x_2) \ldots, \ x_n = f(x_{n-1}), \ldots
\]

We have drawn the “hopping” diagram of the iteration showing the values of \( x_n \) on the \( x \) axis. We have also make a “time series” plot which is produced by graphing the ordered pairs \((n, x_n)\) in two dimensions. Now we will learn a third way to visually study dynamical systems: the Staircase plot.

To create these plots, we first graph the function \( y = f(x) \) that we are using in the iteration. We also graph the line \( y = x \) which we call the diagonal.

First we will learn how to make the staircase plots the “long way”. Then we will learn a quicker way to do the plots.

**Long way to make Staircase Plots:** Follow these instructions using the graph of \( y = f(x) = 2x \). Choose \( x_0 > 0 \) but pretty close to the origin.

Plot the initial value \( x_0 \) on the \( x \) axis. This is the point \((x_0, 0)\) when viewed in two dimensions. Use the graph \( y = f(x) \) to determine the value of \( x_1 = f(x_0) \). Draw a vertical line from \((x_0, 0)\) to the point on the graph \((x_0, x_1 = f(x_0))\). Then draw a horizontal line from this point on the graph to the point \((0, x_1 = f(x_0))\) on the \( y \) axis. Graph the \( x_1 \) value on the \( y \) axis.

Now we would like to get this \( x_1 \) value graphed back on the \( x \) axis. Use the graph of the diagonal: go horizontally from \((0, x_1)\) to the diagonal to get the point \((x_1, x_1)\). Now draw a vertical line from the diagonal to the \( x \) axis to get the point \((x_1, 0)\).

Now repeat what you have done to find the point \( x_2 = f(x_1) \) on the \( y \)-axis by going first vertically to the function and then horizontally to the \( y \)-axis. Plot \( x_2 \) on the \( y \) axis. Then plot \( x_2 \) back on the \( x \) axis: go horizontally from the \( y \)-axis to the diagonal and then vertically from the diagonal to the \( x \)-axis. You will end up plotting the point \((x_2, 0)\) on the \( x \) axis.

Repeat the procedure to plot \( x_3, x_4, \ldots \).

**Quick way to make Staircase Plots:** Again use the graph \( y = f(x) = 2x \) and choose the same initial value \( x_0 \) as before.

Note that in the above method, you go from the graph of the function \((x_0, x_1 = f(x_0))\) to the \( y \)-axis \((0, x_1)\) via a horizontal line. Then at the next stage you go from the \( y \)-axis to the diagonal via another horizontal line. Let us cut out this intermediate steps and now
go directly from the graph of the function at \((x_0, x_1 = f(x_0))\) to the diagonal \((x_1, x_1)\) via a horizontal line.

Then from the diagonal, you went vertically down to the x-axis at \((x_1, 0)\) and then vertically back up to the function \((x_1, f(x_1))\). Again let us cut out the intermediate step and go vertically from the diagonal \((x_1, x_1)\) to the function \((x_1, f(x_1) = x_2)\).

Now repeat; function \(\rightarrow\) horizontal \(\rightarrow\) diagonal \(\rightarrow\) vertical \(\rightarrow\) function \(\rightarrow\) repeat \(\rightarrow\)

Once you have gotten the hang of how to do the staircase iteration, see what happens to iterations for different functions:

\[
f(x) = \frac{1}{2}x, \quad f(x) = -\frac{1}{2}x, \quad f(x) = -2x.
\]