Existence and Uniqueness of Solutions

1. Solve: $x^2 + 1 = 0, \quad x^2 - 1 = 0, \quad x^2 = 0$.

Does the solution exist? If the solution exists, is it unique?

2. Solve the quadratic equation

$$ax^2 + bx + c = 0.$$ 

Under what conditions will there exist a real solution? Under what conditions will there exist a unique real solution.

Theorem 1: If

then there exists a real solution of the quadratic equation

$$ax^2 + bx + c = 0$$

Theorem 2: If

then there exists a unique real solution of the quadratic equation

$$ax^2 + bx + c = 0.$$ 

Note that for uniqueness to hold, the condition that insures existence must first hold but then a bit more is required.
3. Consider the autonomous differential equation \( \frac{dy}{dt} = f(y) \). What are conditions that will insure the existence of equilibrium solutions? What conditions will insure that there exists a unique equilibrium solution?

**Existence Theorem:** Let \( f(y) \) be a continuous function for \( y \in [-10, 10] \). If \( f(-10) > 0 \) and \( f(10) < 0 \) then there exists an equilibrium solution \( y(t) \equiv c \) for some value of \( c \in (-10, 10) \).

**Proof:**

Note: There may be one solution or many depending on the particular function \( f(y) \) involved.

**Uniqueness Theorem:** Let \( f(y) \) be a continuous function for \( y \in [-10, 10] \). if \( f(-10) > 0 \) and \( f(10) < 0 \) and \( f'(y) < 0 \) for all \( y \in [-10, 10] \) then there exists an equilibrium solution \( y(t) \equiv c \) for some value of \( c \in (-10, 10) \).

**Proof:**

Note: There is a unique solution but we do not know what it is!
Informal Existence Theorem The differential equation $\frac{dy}{dt} = f(t, y)$ will have a solution that satisfies the initial condition $y(t_0) = y_0$ providing that $f(t, y)$ is continuous.

Example: $\frac{dy}{dt} = \sin(t^2)$ with initial condition $y(t = \pi/2) = 1$. Does there exist a solution? What is it?