Sequences and Limits

GROUP MEMBERS:

1. ______________________________

2. ______________________________

Problem: Examine sequences and limits numerically and graphically.

A sequence is an infinite list of numbers.

Ex. 1, 1/2, 1/3, 1/4 , 1/5, …..
2, 4, 6, 8, 10, 12, ….

More abstractly, we write a sequence a_1, a_2, a_3, a_4, …, a_n, ….

1. In the above example: a_1 =1 , a_2 = 1/2, a_3=1/3, a_4= 1/4. Do you recognize a pattern?

The value of a_{10} =

What is the general rule for the sequence? a_n =

2. For the second sequence above, give the values:

a_1 =
a_2 =
a_3 =
a_4 =

The value of a_{10} =

What is the general rule for the sequence? a_n =

3. Make up a sequence of numbers. Write down the first 5 numbers in the sequence. Also, describe what the general rule for the sequence is; either in words or using a formula for a_n.

4. Using the Excel spreadsheet program in the Applications folder, enter the values for the first sequence as Professor Donnay will demonstrate. Take the first 25 values in the sequence. Then use Excel to make a graph (use the "line" plot) of the sequence with the index number of the sequence on the horizontal axis and the value of the sequence element on the vertical axis.

Limit of a sequence: By examining the graph, what would you say is the value of the following limit:

$$\lim_{n \to \infty} a_n =$$

Let's denote the limit value by the symbol l. Do the values in the sequence exactly reach l? If not, how long in the sequence must you wait until the values in the sequence get close to l?

5. Dynamical Systems and Iteration (Composition of Functions). An important way to generate sequences of numbers is via iteration (or composition of functions).

Ex.
$a_1 = 1$
$a_2 = 2\,a_1 = 2\,(1) = 2$
$a_3 = 2a_2 = 2(2) = 4$
$a_4 = 2a_3 = 2(4) = 8$
.
.
.
$a_n = 2\,a_{n-1}$

Each number is generated from the number before it by a rule: in this case the doubling rule. We are iterating (or repeating) the rule.

In science, you might be studying some type of system (springs in physics, chemical reaction, the weather, the earth's ecosystem). You know what the present state of the system is and you use the rule of the system (Newtonian mechanics, chemical dynamics, meteorology, global climate modeling) to predict the state of the system one time step (a second, a day, a month, a year, 10 years) in the future. You then reapply the rule to this new value, getting the next value. You would like to know far in the future, how the system will behave. Mathematically, you are again asking for $\lim_{n \to \infty} a_n$.

For the sequence gotten by the doubling rule, what is $\lim_{n \to \infty} a_n =$?

6. Iterate the Cos(x) function 25 times using Excel and make a graph:

$a_1 = \mathrm{Cos}(.2)$
$a_2 = \mathrm{Cos}(a_1) = \mathrm{Cos}(\mathrm{Cos}(.2))$
$a_3 = \mathrm{Cos}(a_2) = \mathrm{Cos}(\mathrm{Cos}(\mathrm{Cos}(.2)))$

$a_4 = \text{Cos}(a_3)$.

For the sequence gotten by the Cos rule, what is $\lim_{n\to\infty} a_n =$?

7. Sometimes the sequence of numbers generated by iterating a rule has a simple looking behaviour. Other times the sequence of numbers, even though it is generated by a rule, does not seem to follow a simple pattern. Then we say the iteration is chaotic.

Ex. The rule of iteration is adding sqrt(2) to the previous number and then taking the resulting number mod 1.

a_1= 0+ sqrt(2) mod 1 = 1.4142 mod 1 = .4142
$a_2 = a_1$ + sqrt(2) mod 1 = .4142 + 1.4142 mod 1 = 2.8282 mod 1 = .8282

.
.

$a_n = a_{n-1}$ + Sqrt(2) mod 1

The Excel command for this is Mod(a_{n-1} + sqrt(2), 1).

Take the first 25 values for the sequence. Plot this sequence using the "line" plot.

a. Does this sequence seem to have a limit?

b. We now make a different type of plot of the sequence. Each number in the sequences lies between 0 and 1. We will plot the values of the sequence on the x axis in the interval [0,1]. For this graph, we will not graph the index of the element of the sequence.

To make the graph work, add a third column to your worksheet. Label the column "y values" and enter 0 for each row. Then highlight both the 2nd and 3rd columns and plot using the "XY (Scatter)" plot. The program is plotting the ordered pair (a_n, 0).

Now increase the number of points in the sequence to 50 and look at the new graph.

Repeat with 75 points.

Repeat with 100 points.

As you take more and more numbers from the sequence and plot them in the interval [0,1], what seems to be happening?

8. In Excel, calculate the first 25 numbers in the sequence a_1= .9 and then $a_n = (a_{n-1})^2$ for n =2, 3, 4, …. Then plot them with "line plot" and predict the $\lim_{n\to\infty} a_n$. Print out your table and graph. To raise a number to a power in Excel, for example a square, you type "^2".

9. Repeat for the sequence $a_n = (1 + 1/n)^n$

10. Repeat for the sequence $a_n = n \text{ Sin}(1/n)$.

11. Repeat for the sequence you made up in problem 3.