1. Finish the definitions of the following concepts on \( R \) in two ways. First with absolute value notation and then with the distance function notation. Recall for \( x, y \in R, d(x, y) = |x - y| \).

a. A sequence \((x_n)\) converges to a limit \( l \) if

b. A sequence \((x_n)\) is Cauchy if

c. A function \( f : R \to R \) is continuous if . . . . Use the \( \delta - \epsilon \) definition.

d. The \( \epsilon \)–neighborhood of a point \( a \in R \) is the set \( V_\epsilon(a) = \ldots \). Make a drawing to illustrate.

2. Now consider \( R^2 \) and the distance function to \( d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \) for \( P_i = (x_i, y_i) \in R^2 \). Starting from what you have above, write out what you think would be reasonable definitions for

a. A sequence \((P_n)\) of points in \( R^2 \) converges to a limit \( l \) if

b. A sequence \((P_n)\) of points in \( R^2 \) is Cauchy if

c. A function \( f : R^2 \to R^2 \) is continuous if . . . . Use \( \delta - \epsilon \) in your definition.

d. The \( \epsilon \)–neighborhood of a point \( a \in R^2 \) is the set \( V_\epsilon(a) = \ldots \). Make a drawing to illustrate.

3. For the following sequences in \( R^2 \), conjecture what their limits are and give a justification for why you think that. Next week, we will learn to make formal proofs for limits in \( R^2 \).

a. \( P_n = (x_n, y_n) = (3 + (-1)^n, -4 - \frac{2}{n}) \).

b. \( Q_n = (x_n, y_n) = (\sin(\frac{1}{n}), 3\cos(\frac{1}{n})) \).

c. \( T_n = (x_n, y_n) = (\frac{4n}{n^2+1}, \frac{2n+n^2}{3n+1}) \).

d. \( S_n = (x_n, y_n) = ((-1)^n, 2 - \frac{1}{n}) \).

e. \( V_n = (x_n, y_n) = (\tan(\frac{1}{n}), n^2) \).

4. Dynamical systems example (Taken from p. 74, Bartle). Consider the sequence produced by starting with the initial point \( x_0 = 8 \) and applying the rule \( x_{n+1} = f(x_n) = \frac{1}{2}x_n + 2 \), for all \( n \in N \).

a. Use Excel to calculate the values \( x_n \) for \( n = 0, 1, 2, \ldots, 20 \). Graph the sequence using Excel. Print out and hand in a copy of your spreadsheet and graph. What is your prediction about the long term behaviour of the iterates?

b. What (important) properties from Real Analysis does this sequence seem to have? After you have answered this question, then turn the page.
c. Prove that this sequence is monotone (using induction).

d. Prove that this sequence is bounded. First give (simple) conjectures for what you think an upper and lower bound would be. Then try to prove that these bounds hold for all \( n \) using induction.

e. Once you have proven that the sequence is monotone and bounded, what can you conclude about the limit? Justify.

f. Find the limit and justify.

5. Use excel to study the following two-dimensional dynamical system. Determine (numerically) what the limit is. Print out your excel spreadsheet and the excel diagram. Initial starting point \( P_0 = (x_0, y_0) = (5, 7) \). Iteration rule is \( P_{n+1} = (x_{n+1}, y_{n+1}) = f(x_n, y_n) \) given by

\[
x_{n+1} = 3 + .9(x_n - 3)\cos(.5) - .9(y_n - 4)\sin(.5) \quad y_{n+1} = 4 + .9(x_n - 3)\sin(.5) + .9(y_n - 4)\cos(.5).
\]