Smooth Approximation of Singular Perturbations

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Dedicated to the Memory of Thoma Pacurar u

Abstract

We consider a certain subclass of self-adjoint extensions of the symmetric operator $-\Delta|C_0^\infty(R - \{S\})$, where $S \subset R$, that correspond formally to perturbations of the Laplacian by potentials involving the $\delta$-potential. We show that these extensions can be approximated in the strong resolvent sense by smooth perturbations of the Laplacian when $S$ is both a finite and infinite subset of $R$. Also, we show that the operator in the finitely-many potential case approaches the operator in the infinitely-many potential case as the number of potentials approaches infinity. These results extend and unify what has previously been known about smooth approximations of point interactions in one dimension.
1. Introduction

The operators that are the main focus of this paper arise as self-adjoint extensions of a symmetric operator \( A \) that include the point interactions in one dimension (cf. [1],[3]). These operators are of mathematical interest because the corresponding models are solvable, to the extent that their resolvents and spectra may be explicitly calculated. In addition, they approximate more realistic models that may be seen empirically. We restrict our attention to one dimension, for in that case the structure is much more interesting than in dimensions two and higher (cf. [14]).

From the theory of self-adjoint extensions (cf. [13]), it follows that the symmetric operator \( A = -\Delta |C_0^\infty(\mathbb{R} - \{0\}) \) on \( L^2(\mathbb{R}) \) has deficiency indices \((2, 2)\), and hence there is a four-parameter family of self-adjoint extensions that are realized formally as perturbations of the Laplacian by polynomials in \( \delta \) and \( \delta' \). Included in these extensions are the operators \(-\Delta + z\delta, -\Delta + z\delta'\), as well as the Fermi pseudo-potentials. The four-parameter family is in one-to-one correspondence with unitary maps between deficiency subspaces. These unitary maps, and hence the self-adjoint extensions, fall into two broad categories. The first, when the unitary matrix is diagonal, completely separates the intervals \((-\infty, 0)\) and \((0, \infty)\). In this case, the operators can be viewed as a direct sum of operators whose terms operate independently on the two intervals (cf. [14]).

Therefore, it is the case when the unitary map is not diagonal that is of interest; the corresponding extensions “link” the intervals \((-\infty, 0)\) and \((0, \infty)\). The class of these exten-
sions, denoted $L_{\alpha,\beta,\gamma,\delta}$, are those whose domains satisfy the boundary conditions

\[
\begin{align*}
    f(0^+) &= \alpha f(0^-) + \beta f'(0^-) \\
    f'(0^+) &= \gamma f(0^-) + \delta f'(0^-)
\end{align*}
\]  

(1)

where $\alpha = a\omega, \beta = b\omega, \gamma = d\omega$ and $\delta = d\omega$, $\omega$ is a complex number of norm one and $a, b, c, d$ are real numbers such that $ad - bc = 1$ ([cf. [14]]). These last conditions guarantee the self-adjointness of the operator. The domains of these extensions consist of all $H^2(\mathbb{R} - \{0\})$ functions satisfying (1), and the corresponding operator is $L_{\alpha,\beta,\gamma,\delta}\phi = -\phi''$. These extensions are studied in depth in [3] and [7], where we calculated the Cayley transform parameterization for this entire class of operators.

One of the widely-studied problems in this context is the approximation of these operators by smooth perturbations of the Laplacian. Intuitively, since $\delta$ can be viewed as a limit of smooth functions in the distribution sense, it is natural to expect that $-\Delta + z\delta$ can be approximated in a suitable sense by $-\Delta + zh_n$, where $h_n$ are smooth functions such that $h_n \rightarrow \delta$. The approximation question has been answered for the $\delta$-potential in the finitely and infinitely-many center cases, (cf. [1, II.2.2, III.1.2]) and for a subclass of $L_{\alpha,\beta,\gamma,\delta}$ in one-center (cf. [3, Theorem 5.1]). The main goal of our work is the extension of the results in [3] to the case of finitely and infinitely-many centers in the non-relativistic setting. In a future work, we will consider the relativistic case (cf. [2,9]).

In Section 2, we review the one-center results from [3], and in Section 3, we introduce
the finitely-many center case. Our main focus is a class of self-adjoint extensions of \( A = -\Delta|C^\infty_0(\mathbb{R} - \{x_1, \ldots, x_k\}) \), which we shall denote \( L_{r,s,z} \). These operators (which include the case \( -\Delta + \sum_{i=1}^k z_i \delta(x - x_i) \)) belong to the many-centered incarnation of the class \( L_{\alpha,\beta,\gamma,\delta} \) above, with boundary conditions

\[
\begin{align*}
\alpha_i f(x_i^-) + \beta_i f'(x_i^-) &= f(x_i^+) \\
\gamma_i f(x_i^-) + \delta_i f'(x_i^-) &= f'(x_i^+)
\end{align*}
\]

\( i = 1, \ldots, k \). We introduce a sequence of approximating operators \( L_{r,s,z,n} \) with smooth potentials, and show that in the case that \( r = s \), \( L_{r,r,z,n} \) converges to \( L_{r,r,z} \) in the strong resolvent sense. While this case does not include the \( \delta \)-potentials, we have already pointed out that the problem has been solved in that case (cf. [1, II.2.2, III.1.2]).

In Section 4, we introduce the infinitely-many centered version of \( L_{r,s,z} \) with centers precisely \( \mathbb{Z} \), and its smooth approximation \( L_{r,s,z,n} \). We then prove strong resolvent convergence. In addition, we address a natural question in Section 5, where we prove that \( L_{r,s,z} \), with \( k \)-centers, converges to \( L_{r,s,z} \), with infinitely many centers.

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2. Singular Perturbations of the Laplacian

We now introduce the specific operators that are the focus of this paper. First, we explore in depth the “one-center” case, and then in Section 3 generalize to finitely-many center. We begin with the construction of $L_{r,s,z}$, a subclass of the above extensions of $A$ that consists, at least in a formal sense, of perturbations of $-\Delta$ by distributions. In [3], these extensions are approximated by bona fide perturbations of $-\Delta$ by smooth potentials. First, to motivate the definition, consider the formal operator $T_z = \frac{d}{dx} + z\delta$, where $z \in \mathbb{C}$. As there is some difficulty in defining $T_z$, Segal [15] proposed the following definition: $T_z = e^{-zH(x)} \frac{d}{dx} e^{zH(x)}$, where $H$ is the Heaviside function and $D(T_z) = \{ \phi \in L^2(\mathbb{R}) \mid e^{zH} \phi \in H^1(\mathbb{R}) \}$. $T_z$ is then a closed operator, and we see that in the event $H$ is a smooth function, then $T_z = \frac{d}{dx} + zH'$. Consequently, a reasonable interpretation of $T_z$ when $H$ is the Heaviside function is $\frac{d}{dx} + z\delta$ (in [3], there is a discussion of renormalization of the coupling constant).

Now consider the operators $T_{z,n} = e^{-zH_n(x)} \frac{d}{dx} e^{zH_n(x)}$, where $z \in \mathbb{C}$,

$$H_n(x) = \int_{-\infty}^{x} h_n(y)dy.$$  \hspace{1cm} (2)

and $\{h_n\}$ are smooth, nonnegative functions with $\text{supp}(h_n) \subset [0, \frac{1}{n}]$, and $\int_{-\infty}^{\infty} h_n(x)dx = 1$.

Let $D(T_n) = H^1(\mathbb{R})$, and for $\phi \in D(T_n)$, $T_{z,n} \phi(x) = e^{-zH_n(x)} \frac{d}{dx} e^{zH_n(x)} \phi(x) = \phi'(x) + zH'_n(x) \phi(x)$. It is easy to see that $T_{z,n} \to T_z$ in the sense of strong group, and hence strong resolvent convergence (cf. [3]).

Since we are interested in extensions of $A$, we consider $L_{r,z} = (T_z + rI)^* (T_z + rI) - r^2 I$. 

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for $r \in \mathbb{R}, z \in \mathbb{C}$ Clearly, $L_{r,z}$ is self-adjoint, and is an extension of $A$. The following is proved in [3]:

[3, Theorem 5.1] given $r \in \mathbb{R}$ and $z \in \mathbb{C}$, the sequence of operators $L_{r,z,n}$ converges to $L_{r,z}$ in the strong resolvent sense,

where $L_{r,z,n} = (r + T_{z,n})^*(r + T_{z,n}) - r^2 I$ and $T_{r,z,n} = e^{-zH_n(x)} \frac{d}{dx} e^{zH_n(x)}$, with $H_n(x)$ are defined by (2).

A slightly broader class of operators, defined in [4], is obtained by considering

$$L_{r,s,z} = (T_z + rH(x) + sH(-x))^*(T_z + rH(x) + sH(-x)) - r^2H(x) - s^2H(-x)$$

where $r, s \in \mathbb{R}$. The importance of this class is that it includes the class $L_{r,z}$, as well as $-\Delta + c\delta$; note that these operators are all self-adjoint, and that $L_{r,s,z} = L_{\alpha,\beta,\gamma,\delta}$, with $\alpha = e^{-z}, \beta = 0, \gamma = e^z s - e^{-z}r$ and $\delta = e^z$. Of course, $L_{r,s,z} = T_z^*T_z$, if $r = s = 0$, and $L_{r,s,z} = -\Delta + c\delta$, if $z = 0$ and $c = s - r$. In the event $r = s$, we obtain the operators $L_{r,z}$.

In the next section, we will extend the approximation result for $L_{r,s}$ to the case of multiple “centers.”

3. Smooth Approximations - Finitely-Many Potentials

We consider the operator $A = -\Delta|_{C^\infty_0(\mathbb{R} - \{x_1, x_2, \ldots, x_k\})} \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}, k \in \mathbb{Z}^+$. $A$ is symmetric, and has deficiency indices $(2k, 2k)$ with $e^{in|x-x_i|}, sgn(x -
$x_i e^{in|x-x_i|}$, $i = 1, \ldots, k$, as a basis for the deficiency subspaces, where $\eta = e^{i\xi}$. Moreover, $A^* \phi = -\phi''$ with $D(A^*) = H^2(\mathbb{R} - \{x_1, x_2, \ldots, x_k\})$.

As in the one-center case, there is a $4k^2$-parameter family of self-adjoint extensions of $A$; as above, we define a family of operators $L_{\alpha, \beta, \gamma, \delta} = -\phi''$, with

$$D(L_{\alpha, \beta, \gamma, \delta}) = \{ \phi \in H^2(\mathbb{R} - \{x_1, \ldots, x_k\}) \}$$

$$\phi(x_i^+) = \alpha_i \phi(x_i^-) + \beta_i \phi'(x_i^-), \phi'(x_i^+) = \gamma_i \phi(x_i^-) + \delta_i \phi'(x_i^-),$$

where $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{C}$, and similarly for $\beta, \gamma$ and $\delta$; $\alpha_i = a_i \omega_i, \beta_i = b_i \omega_i, \gamma_i = c_i \omega_i, \delta_i = d_i \omega_i, a_i, b_i, c_i, d_i \in \mathbb{R}, a_i d_i - b_i c_i = 1, \omega_i \in \mathbb{C}, |\omega_i| = 1$, for $i = 1, \ldots, k$.

First, we extend $T_z$ to an operator that formally corresponds to $T_z = \frac{d}{dx} + \sum_{i=0}^{k} z_i \delta_i(x-x_i)$, where $z = \{z_1, z_2, \ldots, z_k\} \subset \mathbb{C}$. We will write $T_z$ rather than $T_{z_1, z_2, \ldots, z_k}$ for convenience.

As in the one-center case, $T_z$ will be defined as $T_z = e^{-\sum_{i=1}^{k} z_i H_i(x)} \frac{d}{dx} e^{\sum_{i=1}^{k} z_i H_i(x)}$, where $H_i(x) = H(x-x_i)$ is just a translated Heaviside function:

$$H_i(x) = \begin{cases} 0 & x \in (-\infty, x_i) \\ 1 & x \in [x_i, \infty) \end{cases},$$

$i = 1, \ldots, k$, and $D(T_z) = \{ \phi \in L^2(\mathbb{R})|e^{\sum_{i=1}^{k} H_i(x)} \phi(x) \in H^1(\mathbb{R}) \}$. The adjoint of $T_z$ is $T_z^* = -e^{\sum_{i=1}^{k} \pi H_i(x)} \frac{d}{dx} e^{-\sum_{i=1}^{k} \pi H_i(x)}$, and $D(T_z^*) = \{ \phi \in L^2(\mathbb{R})|e^{-\sum_{i=1}^{k} \pi H_i(x)} \phi \in H^1(\mathbb{R}) \}$.

We now define $L_{r,s,z}$ by

$$L_{r,s,z} = \left[ T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right]^* \left[ T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right].$$

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\[- \sum_{i=1}^{k} r_i H_i(x) + s_i H_{i-}(x) \] ^2

where \( r = \{ r_1, r_2, ..., r_k \}, s = \{ s_1, s_2, ..., s_k \} \subset \mathbb{R} \), \( H_{i-}(x) = H(x_i - x) \) is the translated Heaviside, rotated about the line \( x = x_i \), and

\[
D(L_{r,s,z}) = \{ \phi \in D(T_2) | [T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))] \phi \in D(T_z^*) \}.
\]

As in the one-center case, will will show that the domain of \( L_{r,s,z} \) is given in terms of boundary conditions.

**Theorem 3.1.** \( L_{r,s,z} = L_{\alpha,\beta,\gamma,\delta} \), with

\[
\alpha_j = e^{-z_j}, \beta_j = 0, \\
\gamma_j = [e^{z_j} (\sum_{i=1}^{j-1} r_i + \sum_{i=j}^{k} s_i) - e^{-z_j} (\sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i)], \text{ and} \\
\delta_j = e^{z_j},
\]

that is,

\[
D(L_{r,s,z}) = \{ \phi \in H^2(\mathbb{R} - \{ x_1, x_2, ..., x_k \}) | \phi(x_j^+) = e^{-z_j} \phi(x_j^-) \},
\]

\[
\phi'(x_j^+) = [e^{z_j} (\sum_{i=1}^{j-1} r_i + \sum_{i=j}^{k} s_i) - e^{-z_j} (\sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i)] \phi(x_j^-) + e^{-z_j} \phi'(x_j^-),
\]

and for \( \phi \in D(L_{r,s,z}) \), \( L_{r,s,z} \phi = -\phi'' \).

**Proof:** Now,

\[
\phi \in D(L_{r,s,z}) = D([T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))] [T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))]
\]

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if and only if $\phi \in D(T_z)$ and

$$[T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))] \phi \in D([T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))]^*) = D(T_z^*) \quad (3)$$

Since $\phi \in D(T_z)$, $e^{\sum_{i=1}^{k} z_i H_i(x)} \phi(x) = \psi(x) \in H^1(\mathbb{R})$. Thus, $\phi(x) = e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi(x)$ so

$$\phi(x_j^+) = e^{-\sum_{i=1}^{j} z_i} \psi(x_j^+)$$

and

$$\phi(x_j^-) = e^{-\sum_{i=1}^{j-1} z_i} \psi(x_j^-)$$

for $j = 1, \ldots, k$. Since $\psi \in H^1(\mathbb{R})$, $\psi(x_j^+) = \psi(x_j^-)$. This yields the boundary conditions

$$\phi(x_j^+) = e^{-z_j} \phi(x_j^-) \quad (4)$$

for all $j = 1, 2, \ldots, k$.

We will obtain another boundary condition from (3). To see this, let $x \neq x_i$, then, using the fact that $H_i'(x) = 0$ for $x \neq x_i$,

$$[T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x))] \phi(x)$$

$$= e^{-\sum_{i=1}^{k} z_i H_i(x)} \frac{d}{dx} (e^{\sum_{i=1}^{k} z_i H_i(x)} \phi(x)) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \phi(x)$$

$$= e^{-\sum_{i=1}^{k} z_i H_i(x)} \left( \sum_{i=1}^{k} z_i H'_i(x) e^{\sum_{i=1}^{k} z_i H_i(x)} \phi(x) + e^{\sum_{i=1}^{k} z_i H_i(x)} \phi'(x) \right)$$

$$+ \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \phi(x)$$

$$= \phi'(x) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \phi(x).$$
Since $\phi(x) = e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi(x)$, and $\psi \in H^1(\mathbb{R})$, we have $\phi'(x) = -\sum_{i=1}^{k} z_i H'_i(x) e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi(x) + e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi'(x)$, and thus $\phi'(x) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x)) \phi(x) = e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi'(x) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x)) \phi(x)$. Since

$$[T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x))] \phi(x) \in D(T_z^*) = \{ \phi \in L^2(\mathbb{R}) | e^{-\sum_{i=1}^{k} \pi H_i(x)} \phi \in H^1(\mathbb{R}) \},$$

we have

$$e^{-\sum_{i=1}^{k} \pi H_i(x)} [T_z + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x))] \phi(x) = e^{-\sum_{i=1}^{k} \pi H_i(x)} \left[ e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi'(x) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x)) \phi(x) \right]$$

$$= \theta(x)$$

for some $\theta \in H^1(\mathbb{R})$. Thus,

$$e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi'(x) + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_i_-(x)) \phi(x) = e^{\sum_{i=1}^{k} \pi H_i(x)} \theta(x).$$

So for each $j$, we find that

$$e^{-\sum_{i=1}^{j+1} z_i \psi'(x_j^+) + \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i} \phi(x_j^+) = e^{\sum_{i=1}^{j+1} \pi \theta(x_j^+)} \tag{5}$$

and

$$e^{-\sum_{i=1}^{j} z_i \psi'(x_j^-) + \sum_{i=1}^{j-1} r_i + \sum_{i=j}^{k} s_i} \phi(x_j^-) = e^{\sum_{i=1}^{j} \pi \theta(x_j^-)} \tag{6}$$

Now, $\theta \in H^1(\mathbb{R})$ implies that $\theta(x_j^+) = \theta(x_j^-)$. Since $\psi(x) = e^{\sum_{i=1}^{k} z_i H_i(x)} \phi(x)$, and $H'_i(x) = 0$ for $x \neq x_i$, we have $\psi'(x_j^+) = e^{\sum_{i=1}^{j} z_i \phi'(x_j^+)}$, and $\psi'(x_j^-) = e^{\sum_{i=1}^{j} z_i \phi'(x_j^-)}$. 
Substituting into (5) and (6), we obtain
\[
\phi'(x^+_j) + \left( \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i \right) \phi(x^+_j) = e^{z_j} \left[ \phi'(x^-_j) + \left( \sum_{i=1}^{j-1} r_i + \sum_{i=j+1}^{k} s_i \right) \phi(x^-_j) \right].
\]

Combining this result with (11) gives the boundary condition
\[
\phi'(x^+_j) = [e^{z_j} \left( \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i \right) - e^{-z_j} \left( \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i \right)] \phi(x^-_j) + e^{z_j} \phi'(x^-_j).
\]

Moreover, \( \phi \in H^2(\mathbb{R} - \{x_1, \ldots, x_k\}) \), for \( \phi(x) = e^{-\sum_{i=1}^{k} z_i H_i(x)} \psi(x) \), with \( \psi \in H^1(\mathbb{R}) \), so we see that \( \psi' \in H^1(\mathbb{R} - \{x_1, \ldots, x_k\}) \). Therefore \( \phi \in H^2(\mathbb{R} - \{x_1, \ldots, x_k\}) \). Thus, we have shown that:

\[
D(L_{r,s,z}) \subset \{ \phi \in H^2(\mathbb{R} - \{x_1, x_2, \ldots, x_k\}) | \phi(x^+_j) = e^{-z_j} \phi(x^-_j) \},
\]

\[
\phi'(x^+_j) = [e^{z_j} \left( \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i \right) - e^{-z_j} \left( \sum_{i=1}^{j} r_i + \sum_{i=j+1}^{k} s_i \right)] \phi(x^-_j) + e^{z_j} \phi'(x^-_j), \quad j = 1, \ldots, k.
\]

It remains to show that, for \( \phi \in D(L_{r,s,z}) \) \( L_{r,s,z} \phi = -\phi'' \). Let \( \phi \) be so defined. Then, for \( x \neq x_i \),

\[
L_{r,s,z} \phi(x) = \left[ T_x + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right] \left[ T_x + \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right] \phi(x)
\]

\[
- \left[ \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right]^2 \phi(x)
\]

\[
= T_x^* T_x \phi(x) + T_x^* \left[ \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right] \phi(x)
\]

\[
+ \left[ \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right] T_x \phi(x)
\]

\[
= \left( e^{\sum_{i=1}^{k} \pi H_{i,n}} \left( -\frac{d}{dx} \right) e^{-\sum_{i=1}^{k} \pi H_{i,n}} e^{-\sum_{i=1}^{k} z_i H_{i,n}} \frac{d}{dx} e^{\sum_{i=1}^{k} z_i H_{i,n}} \phi(x) \right)
\]

\[
+ \left( e^{\sum_{i=1}^{k} \pi H_{i,n}} \left( -\frac{d}{dx} \right) e^{-\sum_{i=1}^{k} \pi H_{i,n}} \left[ \sum_{i=1}^{k} (r_i H_i(x) + s_i H_{i-}(x)) \right] \phi(x) \right)
\]

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\[ + \sum_{i=1}^{k} \left( r_i H_i(x) + s_i H_i^{-}(x) \right) e^{-\sum_{i=1}^{k} z_i H_i n} \frac{d}{dx} e^{\sum_{i=1}^{k} z_i H_i n} \phi(x) \]

\[ = -\phi''(x) - \sum_{i=1}^{k} \left( r_i H_i(x) + s_i H_i^{-}(x) \right) \phi'(x) + \sum_{i=1}^{k} \left( r_i H_i(x) + s_i H_i^{-}(x) \right) \phi'(x) \]

\[ = -\phi''(x), \]

since all derivatives of \( H_i \) vanish away from \( x_i \) for all \( i \). Thus \( L_{r,s,z} \phi = -\phi'' \) and \( L_{r,s,z} \subset L_{\alpha,\beta,\gamma,\delta} \), so the two are equal. 

We now return to the problem of smooth approximations. In [3, Theorem 5.1], it is shown that the subclass of operators \( L_{r,z} \) can be approximated in the sense of strong resolvent convergence by perturbations of the Laplacian by smooth potentials in the one-center case.

In this section, we show that this result can be extended to the case of finitely-many centers (Theorem 5.2). We note that when \( r_i = s_i \) for all \( i \), that \( \sum_{i=1}^{k} \left( r_i H_i(x) + s_i H_i^{-}(x) \right) = \sum_{i=1}^{k} \left( r_i H_i(x) + r_i H_i^{-}(x) \right) = \sum_{i=1}^{k} r_i I \). This sum will be denoted simply by \( rI \). The domain of \( L_{r,z} \) in this special case is given by

\[ D(L_{r,z}) = \{ \phi \in H^2(\mathbb{R} - \{ x_1, x_2, \ldots, x_k \}) | \phi(x^+) = e^{-z_j} \phi(x^-), \]

\[ \phi'(x^+) = [re^{z_j} - re^{-z_j}] \phi(x^-) + e^{z_j} \phi'(x^-), j = 1, \ldots, k \}. \]

For the approximating operators, define \( T_{z,n} = e^{-\sum_{i=1}^{k} z_i H_i n} \frac{d}{dx} e^{\sum_{i=1}^{k} z_i H_i n} \), where \( H_i n(x) = \int_{-\infty}^{x} h_i n(y)dy \) with \( h_i n \) absolutely continuous, nonnegative functions with support in \([ x_i, x_i + \frac{1}{n} ] \), such that \( \int_{-\infty}^{\infty} h_i n(x) dx = 1 \) for each \( i, n \).
We define $L_{r,z,n} = [T_{z,n} + rI][T_{z,n} + rI] - r^2 I$. We note that

$$L_{r,s,z,n} = [-e\sum_{i=1}^{k} z_i H_{i,n} \frac{d}{dx} e^{-\sum_{i=1}^{k} z_i H_{i,n}} + rI][e^{-\sum_{i=1}^{k} z_i H_{i,n}} \frac{d}{dx} e^{\sum_{i=1}^{k} z_i H_{i,n}} + rI] - r^2 I,$$

and that $D(L_{r,z,n}) = H^2(\mathbb{R})$.

**Theorem 3.2.** Given $r = \{r_1, ..., r_k\} \subset \mathbb{R}, z = \{z_1, ..., z_k\} \subset \mathbb{C}$, the sequence of operators $L_{r,z,n}$ converges to $L_{r,z}$ in the strong resolvent sense.

Before we prove Theorem 3.2, we will need the following result, whose straightforward proof we omit.

**Lemma 3.3.** $e^{-\sum_{i=1}^{k} z_i H_{i,n}(x)}$ converges to $e^{-\sum_{i=1}^{k} z_i H_{i}(x)}$ in the strong operator topology. In addition, $e\sum_{i=1}^{k} z_i H_{i,n}(x)$ converges to $e\sum_{i=1}^{k} z_i H_{i}(x)$ in the strong operator topology.

**Proof of Theorem 3.2:** We will prove the theorem by considering two cases. In the first case, we assume that $r \neq 0$, (i.e., at least one $r_i \neq 0$) and in the second that $r = 0$. In both cases, however, we will employ some of the theory of strongly continuous semigroups. The operator $T_z$ generates a strongly continuous one-parameter group of operators on $L^2(\mathbb{R})$ given by $e^{tT_z} = e^{-\sum_{i=1}^{k} z_i H_{i}(x)} e^{tD} e^{\sum_{i=1}^{k} z_i H_{i}(x)}$, where $e^{tD} = U_t$ is the translation group on $L^2(\mathbb{R})$ generated by $D = \frac{d}{dx}$ with $Dom(D) = H^1(\mathbb{R})$ [10, Ex IX.1.9]. That is, for $\phi \in L^2(\mathbb{R})$,

$$(e^{tT_z}\phi)(x) = (e^{-\sum_{i=1}^{k} z_i H_{i}(x)} e^{tD} e^{\sum_{i=1}^{k} z_i H_{i}(x)} \phi)(x) = e^{-\sum_{i=1}^{k} z_i H_{i}(x)} e^{\sum_{i=1}^{k} z_i H_{i}(x+t)} \phi(x + t) = e^{\sum_{i=1}^{k} z_i H_{i}(x+t) - \sum_{i=1}^{k} z_i H_{i}(x) - \sum_{i=1}^{k} z_i H_{i}(x)} \phi(x + t).$$

Now $\|e^{tT_z}\phi\| \leq \|e^{-\sum_{i=1}^{k} z_i H_{i}(x)}\| \|e^{tD}\| \|e^{\sum_{i=1}^{k} z_i H_{i}(x)} \phi\| \leq e^{2\sum_{i=1}^{k} Re z_i} \|\phi\|$, where $\|e^{-\sum_{i=1}^{k} z_i H_{i}}\|$ denotes the operator norm of the multiplication opera-
tor $e^{\sum_{i=1}^{k} z_i H_i(x)}$. Thus $\|e^{T_z \phi}\| \leq e^{2\sum_{i=1}^{k} \Re z_i} \|\phi\|$, and so for all $t > 0$, $\|e^{T_z}f\| \leq e^{2\sum_{i=1}^{k} \Re z_i}$.

The resolvent of $T_z$ can be written as the Laplace transform of the semigroup $e^{T_z}$:

$$(\lambda + T_z)^{-1} f = \int_{0}^{\infty} e^{\lambda t} e^{T_z} f dt, \text{ for } f \in L^2(\mathbb{R}).$$

Thus

$$\|(\lambda + T_z)^{-1} f\| \leq \int_{0}^{\infty} |e^{-\lambda t}| \|e^{T_z} f\| dt \leq e^{2\sum_{i=1}^{k} \Re z_i} \|f\| \int_{0}^{\infty} |e^{-(\Re \lambda) t}| dt = e^{2\sum_{i=1}^{k} \Re z_i} \|f\| \frac{1}{|\Re \lambda|}.$$ 

so that $\|(\lambda + T_z)^{-1}\| \leq \frac{e^{2\sum_{i=1}^{k} \Re z_i}}{|\Re \lambda|}$, for $\Re \lambda \neq 0$.

Recall that $T^*_z = -e^{\sum_{i=1}^{k} \pi_i H_i(x) \frac{d}{dx}} e^{-\sum_{i=1}^{k} \pi_i H_i(x)} = -T_{\pi}$. So, by a similar process, we derive the same estimates for $e^{T^*_z}$ and $\|(\lambda + T^*_z)^{-1}\|$: $\|e^{T^*_z}\| \leq e^{2\sum_{i=1}^{k} \Re z_i}$, and thus $\|(\lambda + T^*_z)^{-1}\| \leq \frac{e^{2\sum_{i=1}^{k} \Re z_i}}{|\Re \lambda|}$.

Since $e^{T^*_z,n} = e^{-\sum_{i=1}^{k} z_i H_i,n} e^{D} e^{\sum_{i=1}^{k} z_i H_i,n}$, again, $\|e^{T^*_z,n}\| \leq e^{2\sum_{i=1}^{k} \Re z_i}$ and $\|(\lambda + T^*_z,n)^{-1}\| \leq \frac{e^{2\sum_{i=1}^{k} \Re z_i}}{|\Re \lambda|}$ and similarly for $e^{T^*_z}$ and $(\lambda + T^*_z)^{-1}$.

Now, let $\phi \in L^2(\mathbb{R})$; then

$$\|e^{-\sum_{i=1}^{k} z_i H_i,n} e^{D} \sum_{i=1}^{k} z_i H_i,n \phi - e^{-\sum_{i=1}^{k} z_i H_i} e^{D} \sum_{i=1}^{k} z_i H_i \phi\|$$

$$\leq \|e^{-\sum_{i=1}^{k} z_i H_i,n} e^{D} \sum_{i=1}^{k} z_i H_i,n \phi - e^{-\sum_{i=1}^{k} z_i H_i} e^{D} \sum_{i=1}^{k} z_i H_i \phi\|$$

$$+ \|e^{-\sum_{i=1}^{k} z_i H_i,n} e^{D} \sum_{i=1}^{k} z_i H_i \phi - e^{-\sum_{i=1}^{k} z_i H_i} e^{D} \sum_{i=1}^{k} z_i H_i \phi\|$$

$$\leq \|e^{-\sum_{i=1}^{k} z_i H_i,n} e^{D} \sum_{i=1}^{k} z_i H_i,n \phi - e^{-\sum_{i=1}^{k} z_i H_i} e^{D} \sum_{i=1}^{k} z_i H_i \phi\|$$

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By Lemma 3.3, it follows that \( s - \lim_{n \to \infty} e^{tH_n} = e^{tH} \), for all \( t \in \mathbb{R} \). We have already shown that the semigroups generated by \( T \) and \( T_n \) are uniformly bounded by \( e^{2 \sum_{i=1}^{k} \Re z_i} \) and thus satisfy the hypothesis of [10, Theorem IX.2.16]. This theorem then implies, for \( \Re \lambda > 0 \), that \( s - \lim_{n \to \infty} (\lambda + T_n)^{-1} = (\lambda + T)^{-1} \).

Next, we let \( \lambda = r \neq 0 \). We have just shown that \((T_{z,n} + r)^{-1}\) converges strongly to \((T + r)^{-1}\). Similarly, \((T_{z,n}^* + r)^{-1}\) converges strongly to \((T_{z}^* + r)^{-1}\). Since the norms of these resolvents are bounded by \( \frac{e^{2 \sum_{i=1}^{k} \Re z_i}}{|r|} \), we have for any \( \phi \in L^2(\mathbb{R}) \),

\[
\|(T_{z,n} + r)^{-1}(T_{z,n}^* + r)^{-1}\phi - (T + r)^{-1}(T_{z}^* + r)^{-1}\phi\| \\
\leq \|(T_{z,n} + r)^{-1}(T_{z,n}^* + r)^{-1}\phi - (T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi\| \\
+\|(T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi - (T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi\| \\
\leq \|(T_{z,n} + r)^{-1}\phi - (T_{z}^* + r)^{-1}\phi\| \\
+\|(T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi - (T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi\| \\
\leq \frac{e^{2 \sum_{i=1}^{k} \Re z_i}}{|r|}\|(T_{z,n} + r)^{-1}\phi - (T_{z}^* + r)^{-1}\phi\| \\
+\|(T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi - (T_{z,n} + r)^{-1}(T_{z}^* + r)^{-1}\phi\|
\]
Since $s - \lim_{n \to \infty} (T_{z,n}^* + r)^{-1} = (T_z^* + r)^{-1}$ and similarly for $T_{z,n}$ and $T_z$, it follows that

$$s - \lim_{n \to \infty} (T_{z,n} + r)^{-1}(T_{z,n}^* + r)^{-1} = (T_z + r)^{-1}(T_z^* + r)^{-1}.$$  

Because $L_{r,z} = (T_z^* + r)(T_z + r) - r^2 I$ on $L^2(R)$, we have $(L_{r,z,n} + r^2)^{-1} = (T_{z,n} + r)^{-1}(T_{z,n}^* + r)^{-1}$ and so

$$s - \lim_{n \to \infty} (L_{r,z,n} + r^2)^{-1} = s - \lim_{n \to \infty} (T_{z,n} + r)^{-1}(T_{z,n}^* + r)^{-1} = (T_z + r)^{-1}(T_z^* + r)^{-1} = (L_{r,z} + r^2)^{-1}.$$

So we have strong convergence of the resolvents for $\lambda = r^2$. By [10, Cor VIII.1.4] then, we have strong resolvent convergence for all non-real $\lambda$ on $L^2(R)$. Therefore, we have strong resolvent convergence of $L_{r,z,n}$ to $L_{r,z}$ when $r$ is nonzero.

Now let $r = 0$. In this case, $L_{r,z} = B_z = T_z^* T_z$, and $L_{r,z,n} = B_{z,n} = T_{z,n}^* T_{z,n}$. It is more convenient to prove strong graph convergence of $B_{z,n}$ from which strong resolvent convergence follows by [12, Theorem VIII.26].

We proceed in a manner similar to that in [3]. Let $R = U_\psi \cup U_\eta$, where $U_\psi = (x_1 - \frac{3}{2}, x_k + \frac{3}{2})$, and $U_\eta = R - [x_1 - \frac{1}{2}, x_k + \frac{1}{2}]$. By [16, Theorem I.12], there exists a partition of unity, that is there exist functions $\alpha_1, \alpha_2 \in C^\infty(R)$ such that $\alpha_1(x) + \alpha_2(x) = 1$ for all $x \in R$, $\alpha_1(x) \leq 1$ and $\alpha_2(x) \leq 1$ for all $x \in R$, and supp $\alpha_1 \subset U_\psi$, supp $\alpha_2 \subset U_\eta$. Now, let $\phi \in D(B_z)$ and define $\psi = \alpha_1 \phi, \eta = \alpha_2 \phi$. Then supp $\psi \subset U_\psi$, supp $\eta \subset U_\eta$ and $\phi = \psi + \eta$. 

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To prove strong graph convergence, we will show there exists a sequence \( \phi_n = \psi_n + \eta_n \in D(B_{z,n}) \), such that \( \phi_n \to \phi \) and \( B_{z,n}\phi_n \to B_z\phi \). First, we let \( \eta_n = \eta \) for all \( n \). On \( U_\eta \),

\[
B_{z,n} \eta = T^*_{z,n} T_{z,n} \eta
\]

\[
= e^{\sum_{i=1}^k \overline{H_{z,n}}(\overline{H_{i,n}}(D)e^{-\sum_{i=1}^k \overline{H_{i,n}}e^{-\sum_{i=1}^k z_i H_{i,n}}} D e^{\sum_{i=1}^k z_i H_{i,n}} \eta})
\]

\[
= e^{\sum_{i=1}^k \overline{H_{z,n}}(\overline{H_{i,n}}(D)e^{-\sum_{i=1}^k 2Re z_i H_{i,n}}} D e^{\sum_{i=1}^k z_i H_{i,n}} \eta)
\]

\[
= e^{\sum_{i=1}^k \overline{H_{z,n}}(\overline{H_{i,n}}(D)e^{-\sum_{i=1}^k \overline{H_{i,n}}} \eta)}
\]

\[
= e^{\sum_{i=1}^k \overline{H_{z,n}} \sum_{i=1}^k \overline{H_{i,n}} e^{-\sum_{i=1}^k \overline{H_{i,n}} \eta} + \sum_{i=1}^k \overline{H_{i,n}} \eta'}
\]

\[
= e^{-\sum_{i=1}^k \overline{H_{z,n}} \sum_{i=1}^k \overline{H_{i,n}} \eta} + \sum_{i=1}^k \overline{H_{i,n}} \eta' + \eta''
\]

\[
= \left( \sum_{i=1}^k \overline{H_{i,n}} \right) \left( \sum_{i=1}^k \overline{H_{i,n}} \right) \eta + \left( \sum_{i=1}^k \overline{H_{i,n}} \right) \eta'
\]

\[
= \left( \sum_{i=1}^k \overline{H_{i,n}} \right) \eta - \left( \sum_{i=1}^k \overline{H_{i,n}} \right) \eta' - \eta''
\]

So for each \( i \) and \( x \in U_\eta - U_\psi \), \( H_{i,n}'(x) = h_{i,n}(x) = 0 \); indeed, either \( x < x_1 - \frac{3}{2} < x_1 \) or \( x > x_k + \frac{3}{2} > x_k + 1 \) in which case \( h_{i,n}(x) \eta = 0 \) since supp \( h_{i,n} \subset [x_i, x_i + \frac{1}{n}] \) for all \( n \).

If \( x \in U_\eta \cap U_\psi \), then for each \( i \) and all \( n \geq 3 \), \( H_{i,n}'(x) = h_{i,n}(x) = 0 \); since either \( x \leq x_1 - \frac{1}{2} \leq x_1 \) or \( x \geq x_k + \frac{1}{2} > x_k + \frac{1}{n} \) for \( n \geq 3 \), in which case \( h_{i,n}(x) \eta = 0 \) since supp \( h_{i,n} \subset [x_i, x_i + \frac{1}{n}] \).

When \( x \in \mathbb{R} - U_\eta \), then \( \eta = 0 \) since supp \( \eta \subset U_\eta \). Hence for all \( n \geq 3 \),

\[
B_{z,n} \eta = -\eta'' = B_z \eta.
\]
Now let $\tilde{D} = \frac{d}{dx}$ on $L^2[-N, N]$ where $N$ is such that $N > \max\{|x_1| - \frac{3}{2}, |x_k| + \frac{3}{2}\}$, with

$$(\tilde{D}) = \{ f \in L^2[-N, N]| f \text{ is absolutely continuous, } f' \in L^2(\mathbb{R}) \text{ and } f(-N) = 0 \}.$$ 

Then $\tilde{D}^* = -\frac{d}{dx}$, with

$$(\tilde{D}^*) = \{ f \in L^2[-N, N]| f \text{ is absolutely continuous, } f' \in L^2(\mathbb{R}) \text{ and } f(N) = 0 \}.$$ 

Both $D$ and $D^*$ generate contraction semigroups on $L^2[-N, N]$. Let $\tilde{T}_z = e^{-\sum_{i=1}^k z_i H_i} \tilde{D} e^{\sum_{i=1}^k z_i H_i}$ on $L^2[-N, N]$, and $\tilde{B}_z = \tilde{T}_z^* \tilde{T}_z$. Now, $\text{supp}\psi \subset [-N, N]$ so let $\tilde{\psi}$ be the restriction of $\psi$ to $[-N, N]$. Then, since $\psi = \alpha_1 \phi \in D(B_z)$, $\tilde{\psi} \in D(\tilde{B}_z)$ and $\tilde{B}_z \tilde{\psi} = B_z \psi$ on $[-N, N]$. We now define the sequence of operators $\tilde{T}_{z,n} = e^{-\sum_{i=1}^k z_i H_i} \tilde{D} e^{\sum_{i=1}^k z_i H_i}$ and $\tilde{B}_{z,n} = \tilde{T}_{z,n}^* \tilde{T}_{z,n}$. 

By a similar argument to the case when $r \neq 0$, we can find bounds for the semigroups generated by $\tilde{T}_z, \tilde{T}_{z,n}, \tilde{T}_z^*$ and $\tilde{T}_{z,n}^*$ and their resolvents. Thus, $s - \lim_{n \to \infty} e^{\tilde{T}_{z,n}} = e^{\tilde{T}_z}$ and, since $\tilde{T}_{z,n}$ and $\tilde{T}_{z,n}^*$ have empty spectra [cf. Ka Problem III.6.8] on $L^2[-N, N]$, they also satisfy the hypothesis of [Ka Theorem IX.2.16] for $\lambda = 0$. Thus,

$$(\tilde{B}_z)^{-1} = \tilde{T}_z^{-1} \tilde{T}_z^* = s - \lim_{n \to \infty} (\tilde{T}_{z,n} \tilde{T}_{z,n}^*) = s - \lim_{n \to \infty} (\tilde{B}_{z,n})^{-1}.$$ 

Again, we have strong convergence of the resolvents for $\lambda = 0$. As before, by [10, Cor VIII.1.4], we have strong resolvent convergence on $L^2[-N, N]$. Thus on $L^2[-N, N]$ we have strong graph convergence by Theorem 2.3. By definition, there exists $\tilde{\psi}_n \in D(\tilde{B}_{z,n})$ such that $\tilde{\psi}_n \to \tilde{\psi}$ and $\tilde{B}_{z,n} \tilde{\psi}_n \to \tilde{B}_z \tilde{\psi}$ in $L^2[-N, N]$. 

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Let $\psi_n = \tilde{\psi}_n$ on $[-N, N]$ and $\psi_n = 0$ otherwise. So $\psi_n \in D(B_{\frac{1}{z}, n})$, $\psi_n \rightarrow \psi$ and

$$B_{\frac{1}{z}, n} \psi = \tilde{B}_{\frac{1}{z}, n} \tilde{\psi}_n \rightarrow \tilde{B}_{\frac{1}{z}} \tilde{\psi} = B_{\frac{1}{z}} \psi \text{ in } L^2(\mathbb{R}).$$

(8)

By (7) and (8), $B_{\frac{1}{z}, n} \phi_n \rightarrow B_{\frac{1}{z}} \phi$, so $B_{\frac{1}{z}, n} \rightarrow B_{\frac{1}{z}}$ in the sense of strong graph convergence, and thus in the sense of strong resolvent convergence. ■

4. Smooth Approximations - Infinitely-Many Potentials

Attention will now be shifted to the case of an infinite number of centers. In our consideration, the centers will be precisely the set of integers, $\mathbb{Z}$. The set may of course be more general than the integers; in [1, III.2] the only restriction on the set of centers (a subset of $\mathbb{R}$) is that the infimum of the distance between adjacent centers is strictly greater than zero.

As before, we consider the operator

$$A = -\Delta |C^\infty_0(\mathbb{R} - \{\ldots, x_{-1}, x_0, x_1, \ldots\}), \text{ where } \{\ldots, x_{-1}, x_0, x_1, \ldots\} = \mathbb{Z}.$$ 

As in the case of finitely many centers, $A$ is symmetric with deficiency indices $(\infty, \infty)$ (cf. [1, p. 254]). In addition, $A^* = -\Delta$ with $D(A^*) = H^2(\mathbb{R} - \{\ldots, x_{-1}, x_0, x_1, \ldots\})$.

We have an infinite-parameter family of self-adjoint extensions of $A$. We give the analogous definition of $L_{\alpha, \beta, \gamma, \delta}$ by

$$D(L_{\alpha, \beta, \gamma, \delta}) = \{\phi \in H^2(\mathbb{R} - \{\ldots, x_{-1}, x_0, x_1, \ldots\})|$$

$$\phi(x^+_i) = \alpha_i \phi(x^-_i) + \beta_i \phi'(x^-_i), \phi'(x^+_i) = \gamma_i \phi(x^-_i) + \delta_i \phi'(x^-_i)\},$$

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where \(\alpha = \{\alpha_i\}_{i=-\infty}^{\infty}, \beta = \{\beta_i\}_{i=-\infty}^{\infty}, \gamma = \{\gamma_i\}_{i=-\infty}^{\infty}, \delta = \{\delta_i\}_{i=-\infty}^{\infty} \subset \mathbb{C}; \alpha_i = a_i\omega_i, \beta_i = b_i\omega_i, \gamma_i = c_i\omega_i, \delta_i = d_i\omega_i, a_i, b_i, c_i, d_i \in \mathbb{R}, a_i d_i - b_i c_i = 1, \omega_i \in \mathbb{C}, |\omega_i| = 1, \text{ for } i = ...,-1,0,1,\ldots, \text{ and for } \phi \in D(L_{\alpha, \beta, \gamma, \delta}), L_{\alpha, \beta, \gamma, \delta} \phi = -\phi''.

We now construct the infinitely many center incarnation of \(L_{r,s,z}\). In order that the multiplication operators be bounded, we will alter the definition of \(T_z\) corresponding to the formal operator \(\frac{d}{dx} + \sum_{i=-\infty}^{\infty} z_i \delta(x - x_i)\). Our approach will be to let the coupling constants be \(z_i = \eta_i, \{..., \eta_{-1}, \eta_0, \eta_1, ...\} \subset \mathbb{R}, \text{ where } \sum_{i=-\infty}^{0} \eta_i \text{ converges absolutely. We define}
\[G_\eta(x) = \sum_{i=-\infty}^{\infty} \eta_i H_i(x)\]
where \(H_i\) is the translated Heaviside function \(H_i(x) = H(x - x_i)\). Note that for any \(x_0 \in [x_j, x_{j+1}) \subset \mathbb{R}\), we have \(G_\eta(x_0) = \sum_{i=-\infty}^{\infty} \eta_i H_i(x_0) = \sum_{i=-\infty}^{j} \eta_i < \infty\), by the restrictions placed on \(\{\eta_i\}_{i=-\infty}^{\infty}\). Therefore, \(G_\eta\) is well defined and real-valued. Since the exponents in the multiplication operators in \(T_\eta\) are purely imaginary, those multiplication operators are bounded. We then define
\[T_\eta = e^{-iG_\eta(x)} \frac{d}{dx} e^{iG_\eta(x)},\]
and, thus \(T_\eta^* = -e^{-iG_\eta(x)} \frac{d}{dx} e^{iG_\eta(x)}\). As was the situation with previous cases, this construction of \(T_\eta\) corresponds to the formal operator \(T_\eta = \frac{d}{dx} + \sum_{i=-\infty}^{\infty} \eta_i \delta(x - x_i)\).

In addition, we require that \(\sum_{i=-\infty}^{0} r_i\) and \(\sum_{i=0}^{\infty} s_i\) both converge absolutely. So we have, for \(x_0 \in [x_j, x_{j+1}) \subset \mathbb{R}\), \(\sum_{i=-\infty}^{\infty} (r_i H_i(x_0) + s_i H_{i-}(x_0)) = \sum_{i=-\infty}^{j} r_i + \sum_{i=j+1}^{\infty} s_i < \infty\). Thus \(\sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x))\) is well defined for all \(x \in \mathbb{R}\).
The analogous self-adjoint extension of $A = -\Delta|C^\infty_0(\mathbb{R} - \{\cdots, x_{-1}, x_0, x_1, \cdots\})$ is now given by

$$L_{r,s,\eta} = (T_{\eta} + \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x)))^* (T_{\eta} + \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x)))$$

$$- \left[ \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x)) \right]^2.$$

The following theorem is the infinitely-many centered analogue of Theorem 3.1.

**Theorem 4.1.** $L_{r,s,\eta} = L_{\alpha,\beta,\gamma,\delta}$, with

$$\alpha_j = e^{-n_j}, \beta_j = 0,$$

$$\gamma_j = [e^{\eta_j} \left( \sum_{i=-\infty}^{j-1} r_i + \sum_{i=j}^{\infty} s_i \right) - e^{-\eta_j} \left( \sum_{i=-\infty}^{j} r_i + \sum_{i=j+1}^{\infty} s_i \right)],$$

$$\delta_j = e^{\eta_j};$$

that is,

$$D(L_{r,s,\eta}) = \{ \phi \in H^2(\mathbb{R} - \{\cdots, x_{-1}, x_0, x_1, \cdots\}) | \phi(x_j^+) = e^{-n_j} \phi(x_j^-),$$

$$\phi'(x_j^+) = [e^{\eta_j} \left( \sum_{i=-\infty}^{j-1} r_i + \sum_{i=j}^{\infty} s_i \right) - e^{-\eta_j} \left( \sum_{i=-\infty}^{j} r_i + \sum_{i=j+1}^{\infty} s_i \right)] \phi(x_j^-) + e^{\eta_j} \phi'(x_j^-) \},$$

and for $\phi \in D(L_{r,s,\eta})$, $L_{r,s,\eta} \phi = -\phi''$.

**Proof:** This proof is quite similar to the proof of Theorem 3.1, and we omit the details.

As in section 3, our motivation is to approximate these operators with smooth potentials.

The sequence of approximating operators will be denoted

$$T_{\eta,n} = e^{-iG_{\eta,n}(x)} \frac{d}{dx} e^{iG_{\eta,n}(x)}$$
where \( G_{\eta,n}(x) = \sum_{i=-\infty}^{\infty} \eta_i H_{i,n}(x) \), and \( H_{i,n}(x) \) is defined in Section 3.

We will again consider the special case \( r_i = s_i \) for all \( i \). Thus

\[
\sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-1}(x)) = \sum_{i=-\infty}^{\infty} (r_i H_i(x) + r_i H_{i-1}(x)) = \sum_{i=-\infty}^{\infty} r_i I,
\]

which will be denoted simply by \( rI \); note that \( \sum_{i=-\infty}^{\infty} r_i \) converges absolutely.

**Theorem 4.2.** The sequence of operators

\[
L_{r,\eta,n} = (T_{\eta,n} + r)^* (T_{\eta,n} + r) - r^2 I
\]

converges to \( L_{r,\eta} \) in the strong resolvent sense.

Before Theorem 4.2 is proved, a lemma is needed.

**Lemma 4.3.** Let

\[
B_{\eta} = T_{\eta}^* T_{\eta} = e^{-iG_{\eta}(x)} \left( -\frac{d^2}{dx^2} \right) e^{iG_{\eta}(x)},
\]

where

\[
D(B_{\eta}) = \{ \phi \in L^2(\mathbb{R}) | e^{iG_{\eta}(x)} \phi \in H^2(\mathbb{R}) \}.
\]

Then \( D_0 = \{ \phi \in D(B_{\eta}) | \text{supp } \phi \text{ is compact} \} \) is a core for \( B_{\eta} \).
Proof: Let \( \phi \in D(B_\eta) \). Define \( \phi_n = \alpha_n \phi \) where \( \alpha_n \in C_0^\infty(\mathbb{R}), 0 \leq \alpha_n \leq 1 \),

\[
\alpha_n(x) = \begin{cases} 
\exp\left( \frac{|x+(n+\frac{1}{4})|^2}{|x+(n+\frac{1}{4})|^2-1} \right) & x \in (-n+\frac{3}{4}, -(n + \frac{1}{4})) \\
\exp\left( \frac{|x-(n+\frac{1}{4})|^2}{|x-(n+\frac{1}{4})|^2-1} \right) & x \in (n + \frac{1}{4}, n + \frac{3}{4}) \\
1 & x \in [-n + \frac{1}{4}, n + \frac{1}{4}] \\
0 & x \in (-\infty, -(n + \frac{3}{4})] \cup [n + \frac{3}{4}, \infty) 
\end{cases}
\]

Then \( \{\phi_n\} \subset D_0 \). Now \( ||\phi_n - \phi||^2 = ||\alpha_n \phi - \phi||^2 = \int_{-\infty}^{\infty} |\alpha_n \phi(x) - \phi(x)|^2\,dx \). For any \( x \in \mathbb{R} \), there is an \( N \) sufficiently large such that \( \alpha_N(x) = 1 \) and \( \phi_N(x) = \phi(x) \), so \( \phi_n(x) \to \phi(x) \) pointwise. Also, for each \( x \in \mathbb{R} \), \( |\alpha_n(x) \phi(x) - \phi(x)|^2 < 4|\phi(x)|^2 \in L^1(\mathbb{R}) \), since \( \phi \in L^2(\mathbb{R}) \). By the dominated convergence theorem, \( \lim_{n \to \infty} \int_{-\infty}^{\infty} |\alpha_n \phi(x) - \phi(x)|^2\,dx = 0 \).

In addition,

\[
\left\|B_\eta \phi_n - B_\eta \phi\right\| = \left\| -e^{-iG_\eta} \frac{d^2}{dx^2}(e^{iG_\eta} \phi_n) + e^{-iG_\eta} \frac{d^2}{dx^2}(e^{iG_\eta} \phi) \right\|
\leq \left\|e^{-iG_\eta} \right\| \left\| \frac{d^2}{dx^2}(e^{iG_\eta} \phi_n) - \frac{d^2}{dx^2}(e^{iG_\eta} \phi) \right\|
= \left\| \frac{d^2}{dx^2}(e^{iG_\eta} \alpha_n \phi) - \frac{d^2}{dx^2}(e^{iG_\eta} \phi) \right\|.
\]

Now, since \( \phi \in D(B_\eta) \), \( e^{iG_\eta} \phi \in H^2(\mathbb{R}) \), so

\[
\frac{d^2}{dx^2}(e^{iG_\eta} \alpha_n \phi) = \frac{d}{dx} \frac{d}{dx} (\alpha_n e^{iG_\eta} \phi) = \frac{d}{dx} \left[ \alpha_n' (e^{iG_\eta} \phi) + \alpha_n \frac{d}{dx} (e^{iG_\eta} \phi) \right] = \alpha_n'' (e^{iG_\eta} \phi) + \alpha_n' \frac{d}{dx} (e^{iG_\eta} \phi)
\]
\[
\begin{align*}
\alpha_n^{'} \frac{d}{dx}(e^{iG_n \phi}) + \alpha_n^{''} \frac{d^2}{dx^2}(e^{iG_n \phi}) \\
= \alpha_n^{'} \frac{d^2}{dx^2}(e^{iG_n \phi}) + 2\alpha_n^{'} \frac{d}{dx}(e^{iG_n \phi}) + \alpha_n^{''}(e^{iG_n \phi}).
\end{align*}
\]

So the above approximation becomes
\[
\| \frac{d^2}{dx^2}(e^{iG_n \phi} \alpha_n \phi) - \frac{d^2}{dx^2}(e^{iG_n \phi}) \| \\
= \| \alpha_n^{'} \frac{d^2}{dx^2}(e^{iG_n \phi}) + 2\alpha_n^{'} \frac{d}{dx}(e^{iG_n \phi}) + \alpha_n^{''}(e^{iG_n \phi}) - \frac{d^2}{dx^2}(e^{iG_n \phi}) \| \\
\leq \| \alpha_n^{'} \frac{d^2}{dx^2}(e^{iG_n \phi}) - \frac{d^2}{dx^2}(e^{iG_n \phi}) \| + 2\| \alpha_n^{'} \frac{d}{dx}(e^{iG_n \phi}) \| + \| \alpha_n^{''}(e^{iG_n \phi}) \|.
\]

Since \( \phi \in D(B_n) \), \( \frac{d^2}{dx^2}(e^{iG_n \phi}) \in L^2(R) \), so by a similar dominated convergence argument to the one above, with the dominating function \( |\frac{d^2}{dx^2}(e^{iG_n \phi})|^2 \), the first term of the above sum approaches zero as \( n \to \infty \). The two latter terms are shown to converge to zero by the following dominated convergence argument:
\[
\| \alpha_n^{'} \frac{d}{dx}(e^{iG_n \phi}) \|^2 = \int_{-\infty}^{\infty} |\alpha_n^{'}(x) \frac{d}{dx}(e^{iG_n(x) \phi(x)})|^2 dx.
\]

For each \( x \in R \), there exists an \( N \) such that for all \( n \geq N, \alpha_n(x) = 1 \) so \( \alpha_n^{'}(x) = 0 \).

Thus the sequence converges pointwise to zero. Now by the construction of \( \alpha_n \), for each \( n \), \( \alpha_n^{'} \) is bounded with support in \( [-n - \frac{3}{4}, -n + \frac{1}{4}] \cup [n + \frac{1}{4}, n + \frac{3}{4}] \). Since \( \alpha_n \in C_0^\infty(R) \), \( \alpha_n^{'} \) attains its maximum on the above compact set. Thus, \( |\alpha_n^{'}(x)| \leq C \) for some \( C \in R \).

This bound applies to all \( n \), since \( \alpha_{n+1} \) is just a translation of \( \alpha_n \) along the \( x \)-axis. Thus we
have a dominating expression

$$|\alpha'_n(x) \frac{d}{dx}(e^{iG_\eta(x)} \phi(x))|^2 \leq C^2 |\frac{d}{dx}(e^{iG_\eta(x)} \phi(x))|^2 \in L^1(\mathbb{R})$$

since $e^{iG_\eta} \phi \in H^2(\mathbb{R}) \subset H^1(\mathbb{R})$. So $\|\alpha'_n \frac{d}{dx}(e^{iG_\eta} \phi)\|_2 \to 0$ as $n \to \infty$. A similar argument holds for the third term, since for all $n, \alpha''_n$ is continuous (and thus bounded on the above compact set), and $e^{iG_\eta} \phi \in H^2(\mathbb{R})$. So $\phi_n \to \phi$ and $B_\eta \phi_n \to B_\eta \phi$. Thus $D_0$ is a core for $D(B_\eta)$. ■

**Proof of Theorem 4.2:** The proof for the case when $r \neq 0$ is similar to the proof of Theorem 3.2, since the exponents of the multiplication operators are purely imaginary and thus those operators are bounded. $T_\eta$ generates a one parameter contraction group. Rather than multiplying $e^{tD}$ on the left and right by exponential multiplication operators with bounded exponents, now the translation group is multiplied on the right and left by the unitary operators $e^{\pm iG_\eta}$, respectively. The one significant difference is that in this case the estimates for the semigroup and resolvents change. The estimate for the semigroup is given by $\|e^{tT_\eta} \phi\| \leq \|e^{iG_\eta(x)}\| \|e^{tD}\| \|e^{iG_\eta(x)} \phi\| \leq \|\phi\|$, since $e^{iG_\eta(x)}$ and $e^{G_\eta(x)}$ are unitary; this leads to the bound for the resolvent $\|(\lambda + T_\eta)^{-1}\| \leq \frac{1}{|\text{Re}\lambda|}$. The same estimates hold for the resolvents of the adjoint, the sequence $T_{\eta,n}$, and the sequence $T_{\eta,n}^*$. The rest of the proof when $r \neq 0$ is identical.

Now consider the case where $r = 0$. As before, $L_{r,\eta} = B_\eta = T_\eta^* T_\eta$, and $L_{r,\eta,n} = B_{\eta,n} = T_{\eta,n}^* T_{\eta,n}$. Here strong resolvent convergence will be proved directly for a specific complex
number $\lambda = i$. Strong resolvent convergence for all non-real $\lambda$ follows by [10, Cor VIII.1.4].

Let $\phi \in L^2(\mathbb{R})$ and $\epsilon > 0$ be given. Since $B_\eta$ is self-adjoint, $\text{Ran}(B_\eta + i) = L^2(\mathbb{R})$. Thus $\phi = (B_\eta + i)\psi$ for some $\psi \in D(B_\eta)$. Lemma 6.3 yields a sequence $\{\psi_m\} \subset D_0$ such that for all $m \geq N_1$, $\|\psi_m - \psi\| < \frac{\epsilon}{4}$. In addition, there exists an $N_2$ such that for all $m \geq N_2$, $\|B_\eta \psi_m - B_\eta \psi\| < \frac{\epsilon}{4}$. Fix $m \geq \max\{N_1, N_2\}$. Now

$$\|(B_{\eta,n} + i)^{-1}\phi - (B_\eta + i)^{-1}\phi\| = \|(B_{\eta,n} + i)^{-1}(B_\eta + i)\psi - \psi\|$$

$$\leq \|(B_{\eta,n} + i)^{-1}(B_\eta + i)\psi - (B_{\eta,n} + i)^{-1}(B_\eta + i)\psi_m\|$$

$$+ \|(B_{\eta,n} + i)^{-1}(B_\eta + i)\psi_m - \psi_m\| + \|\psi_m - \psi\|$$

$$\leq \|(B_{\eta,n} + i)^{-1}\| \|B_\eta \psi_m - B_\eta \psi\| + \|\psi_m - \psi\|$$

$$+ \|(B_{\eta,n} + i)^{-1}(B_\eta + i)\psi_m - (B_\eta + i)^{-1}(B_\eta + i)\psi_m\|$$

$$+ \|\psi_m - \psi\|,$$

since $\|(B_{\eta,n} + i)^{-1}\| \leq 1$, for all $n$. Clearly all but the third term in the last expression on the right converge to zero as $n \to \infty$. The argument can be concluded by noting that, since $\psi_m \in D_0$ has compact support, so does $(B_\eta + i)\psi_m$. Hence $(B_\eta + i)\psi_m$ is non-zero over only finitely many centers. Thus, there exists a $J$ such that $x_i \not\in \text{supp} \ (\psi_m)$ for all $|i| > J$. Therefore, the remaining term is non-zero on the bounded set $(x_{-J}, x_J) \subset \mathbb{R}$. On this set $G_\eta = \sum_{i=-J}^{J} \eta_i H_i$ and $G_{\eta,n} = \sum_{i=-J}^{J} \eta_i H_{i,n}$ where $H_i, H_{i,n}$ are as in Theorem 3.2. Letting $z_i = i\eta_i, r = 0$, Theorem 3.2 implies that $\lim_{n \to \infty} (B_{\eta,n} + i)^{-1} = (B_\eta + i)^{-1}$, and so the third
term converges to zero, as well. Consequently, \( L_{r,\eta,n} \) converges to \( L_{r,\eta} \) in the sense of strong resolvent convergence. ■

5. Infinitely-Many Potentials Approximated by Finitely-Many

A natural question to ask is whether the operator in Theorem 3.2 approaches that in Theorem 4.1, as the number of centers tends to infinity. We now wish to prove that \( L_{r,s,\eta} \), defined on finitely many centers converges to \( L_{r,s,\eta} \), defined on infinitely many centers in the strong resolvent sense, as the number of centers approaches infinity. The approach taken will be the same as in [1, Theorem III.2.1.1]. Recall that \( L_{r,s,\eta} \), with infinitely many centers takes the form

\[
L_{r,s,\eta} = [T_\eta + \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x))] [T_\eta + \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x))]^* \]

\[
- \left( \sum_{i=-\infty}^{\infty} (r_i H_i(x) + s_i H_{i-}(x)) \right)^2,
\]

where \( T_\eta = e^{-i \sum_{i=-\infty}^{\infty} \eta H_i(x)} \frac{d}{dx} e^{i \sum_{i=-\infty}^{\infty} \eta H_i(x)} \) and \( T_\eta^* = -e^{-i \sum_{i=-\infty}^{\infty} \eta H_i(x)} \frac{d}{dx} e^{i \sum_{i=-\infty}^{\infty} \eta H_i(x)} \) on the domain

\[
D(L_{r,s,\eta}) = \{ \phi \in H^2(\mathbb{R} - \{\ldots, x_{-1}, x_0, x_1, \ldots\}) | \phi(x^+_j) = e^{-\eta_j} \phi(x^-_j), \}
\]

\[
\phi'(x^+_j) = e^{\eta_j} \left( \sum_{i=-\infty}^{j-1} r_i + \sum_{i=0}^{\infty} s_i \right) - e^{-\eta_j} \left( \sum_{i=-\infty}^{j} r_i + \sum_{i=j+1}^{\infty} s_i \right) \phi(x^-_j) + e^{\eta_j} \phi'(x^-_j),
\]

\[
j = \ldots, -1, 0, 1, \ldots \}.
\]

In this case our approximating operators will be denoted \( L_{r,s,\eta,n} \), where

\[
L_{r,s,\eta,n} = [T_{\eta,n} + \sum_{i=-n}^{n} (r_i H_i(x) + s_i H_{i-}(x))] [T_{\eta,n} + \sum_{i=-n}^{n} (r_i H_i(x) + s_i H_{i-}(x))]^* \]

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\[-\left[ \sum_{i=-n}^{n} r_i H_i(x) + s_i H_i(x) \right]^2\]

where $T_{\eta,n} = e^{-\sum_{i=-n}^{n} \eta_i H_i(x)} \frac{d}{dx} e^{\sum_{i=-n}^{n} \eta_i H_i(x)}$ and $T_{\eta,n}^* = -e^{-\sum_{i=-n}^{n} \eta_i H_i(x)} \frac{d}{dx} e^{\sum_{i=-n}^{n} \eta_i H_i(x)}$

on the domain

$$D(L_{r,s,\eta,n}) = \{ \phi \in H^2(\mathbb{R} - \{x_{-n}, \ldots, x_n\}) | \phi(x_j^+) = e^{-\eta_j} \phi(x_j^-), \phi'(x_j^+) = e^{\eta_j} (\sum_{i=-n}^{j-1} r_i + \sum_{i=j}^{n} s_i) \phi(x_j^-) + e^{\eta_j} \phi'(x_j^-), j = -n, \ldots, n \}.$$ 

**Theorem 5.1.** Given $r_i, s_i, \eta_i \in \mathbb{R}$ as defined above, the sequence of operators $L_{r,s,\eta,n}$ converges to $L_{r,s,\eta}$ in the strong resolvent sense.

**Proof:** Let $D_0 = \{ \phi \in D(L_{s,r,\eta}) | \text{supp } f \text{ is compact} \}$. We claim that $D_0$ is a core for $D(L_{r,s,\eta})$. Let $\phi \in D(L_{r,s,\eta})$ and truncate $\phi$ using $\alpha_n$ as in the proof of Lemma 4.3, so that $\phi_n = \alpha_n \phi$ and $\|\alpha_n'\|_{\infty} + \|\alpha_n''\|_{\infty} \leq c$ for some constant $c$. We need to show that $\phi_n$ and $L_{r,s,\eta,\phi_n}$ converge strongly to $\phi$ and $L_{r,s,\eta,\phi}$, respectively. As in the proof of Lemma 4.3, we have that $\phi_n \to \phi$ in $L^2(\mathbb{R})$.

In addition, we see that $L_{s,r,\eta,\phi_n} = -\Delta (\alpha_n \phi) = -(\alpha_n \phi'' + 2\alpha_n' \phi' + \alpha_n'' \phi)$. Thus $\|L_{r,s,\eta,\phi_n} - L_{r,s,\eta,\phi}\| = \|\alpha_n \phi'' + 2\alpha_n' \phi' + \alpha_n'' \phi - \phi''\|$. Now, since $\alpha_n$ is constant outside $(-n - \frac{3}{4}, -n - \frac{1}{4}) \cup (n + \frac{1}{4}, n + \frac{3}{4})$, $\alpha_n' \to 0$ and $\alpha_n'' \to 0$ pointwise. Since all three $\alpha_n$ terms are bounded above by the $L^2(\mathbb{R})$ functions $\phi''$, $2\phi'$ and $\phi$, respectively, a dominated convergence argument completes the proof. 

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References


