John Bell (1964): What if the two observers make measurements along three non-orthogonal axes?

The two observers use the same set of three axes. Now the hidden variables must be ready for 8 possibilities. In each case, if the two observers make measurements along the same axis, they must get opposite results, to conserve angular momentum.

<table>
<thead>
<tr>
<th>Probability</th>
<th>Particle 1</th>
<th>Particle 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>N₁</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>N₂</td>
<td>++</td>
<td>+</td>
</tr>
<tr>
<td>N₃</td>
<td>+</td>
<td>--</td>
</tr>
<tr>
<td>N₄</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>N₅</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>N₆</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>N₇</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>N₈</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Since probabilities are positive,

\[N₃ + N₄ \leq (N₂ + N₄) + (N₃ + N₇)\]

Probability of measuring \(S_{a₁} = \frac{\hbar}{2}\) and \(S_{b₂} = \frac{\hbar}{2}\)

\[\equiv P(+\hat{a}, +\hat{b})\]

From the chart, \(N₃ + N₄ = P(+\hat{a}, +\hat{b})\)

\(N₂ + N₄ = P(+\hat{a}, +\hat{c})\) \(N₃ + N₇ = P(+\hat{c}, +\hat{b})\)

⇒ Hidden variable theory predicts

\[\hat{P}(+\hat{a}, +\hat{b}) \leq P(+\hat{a}, +\hat{c}) + P(+\hat{c}, +\hat{b})\]

a "Bell's inequality".

What does conventional quantum mechanics predict?

\[P(+\hat{a}, +\hat{b}) = 1\langle +\hat{a}, +\hat{b} | \psi \rangle^2\]

Let's choose \(\theta_{ab} = 120°\), \(\theta_{ac} = 60° = \theta_{cb}\)

\[\rightarrow P(+\hat{a}, +\hat{b}) = \frac{3}{8} > \frac{P(+\hat{a}, +\hat{c}) + P(+\hat{c}, +\hat{b})}{1/8}\]

Thus, this prediction for the direction of \(\psi\) the inequality is opposite that of hidden variable theory.

1982: Aspect et al. perform analogous experiments on photons

⇒ hidden variable theory is wrong!
The position operator

\( \langle x \rangle : \) a position eigenstate, with the particle completely localized at position \( x \)

\[ \hat{x} |x\rangle = x |x\rangle \]

Because \( x \) varies continuously, we must modify the identity operator:

\[ \hat{I} = \sum_{i} |i\rangle \langle i| \rightarrow \hat{I} = \int_{-\infty}^{\infty} dx \ |x\rangle \langle x| \]

\[ \Rightarrow \ 4\rangle = \int_{-\infty}^{\infty} dx \ |x\rangle \langle x|4\rangle \] a generic state, expressed as a superposition of position eigenstates

\[ \psi(x) \equiv \langle x|4\rangle \]

The wavefunction

special case: \( \langle 4 \rangle = |x\rangle \rightarrow |x\rangle = \int_{-\infty}^{\infty} dx \ |x\rangle \langle x|x\rangle \)

\[ \Rightarrow \ \langle x|x' \rangle = \delta(x-x'), \ \text{the "Dirac delta function"} \]

\[ \int_{-\infty}^{\infty} f(x) \delta(x-x') \ dx = f(x') \]

Normalization

\[ \langle 4|4 \rangle = 1 \rightarrow \int_{-\infty}^{\infty} \ |4\rangle \langle 4| \ dx = 1 \]

Expectation values

\[ \hat{A}(x) : \text{an operator that can be computed based on knowledge of } dx \]

\[ \rightarrow \ \langle A(x) \rangle = \int_{-\infty}^{\infty} \ |4\rangle \langle 4| A(x) \ dx \]
The translation operator
\[ \hat{T}(a)|x\rangle = |x+a\rangle, \text{i.e. the state is translated} \]
to the right by \( a \).

The generator of translations
\[ \hat{T}(dx) = 1 - \frac{i}{\hbar} \hat{g} \hat{dx} \]
\( \hat{g} \) - generator of translations

Townsend pp. 153-5 \( \Rightarrow \) [\( \hat{x}, \hat{g} \)] = i\hbar

The momentum operator

The "correspondence principle" states that quantum mechanics must give predictions that are consistent with classical mechanics, when applied to large objects.

\[ \Rightarrow \langle p_x \rangle = m \frac{d \langle x \rangle}{dt} \]

\[ \Rightarrow [\hat{x}, \hat{p}_x] = i\hbar \]

Therefore, we identify the generator of translations \( \hat{g} \) with the momentum operator \( \hat{p}_x \).