Proof that $\psi(\phi=0) = \psi(\phi=2\pi)$

$\hat{L}_z$ is Hermitian $\Rightarrow$ for arbitrary state $|\chi\rangle$, we have $\hat{L}_z^\dagger \hat{L}_z = \hat{L}_z \hat{L}_z^\dagger$ not $x$.

$\langle \psi | \hat{L}_z | \chi \rangle = \langle \chi | \hat{L}_z^\dagger | \psi \rangle^*$

express in $\phi$-basis,
but only show $\phi$-dependence explicitly

\[
\int_0^{2\pi} \psi^*(\phi) \frac{\hbar}{i} \frac{\partial}{\partial \phi} \chi(\phi) \, d\phi = \int_0^{2\pi} \chi^*(\phi) \frac{\hbar}{i} \frac{\partial}{\partial \phi} \psi(\phi) \, d\phi^* \\
\]

LHS
Integrate by parts: $v = \chi \Rightarrow dv = \frac{d\chi}{d\phi} \, d\phi$
$u = \psi^*$

$\Rightarrow$ LHS = $\frac{\hbar}{i} \left[ \psi^* \chi \bigg|_0^{2\pi} - \int_0^{2\pi} \chi \frac{d\psi^*}{d\phi} \, d\phi \right] = -\frac{\hbar}{i} \int_0^{2\pi} \chi \frac{d\psi^*}{d\phi} \, d\phi$

$\Rightarrow \psi^* \chi \bigg|_0^{2\pi} = 0$.

Since $\psi$ and $\chi$ are arbitrary, we must have $\psi^*(\phi=2\pi) = \psi^*(\phi=0)$

take adjoint

$\psi(\phi=2\pi) = \psi(\phi=0)$