#1) Bye bye Snowball

All of the snowball's momentum goes into the Earth, which then translates (and rotates) a tiny bit faster (or slower, depending on which way the ball was thrown).

What is the change in kinetic energy of the Earth?

From momentum conservation:

Initial \( p \) = Final \( p \)

\[
\frac{mv}{(m+M)} = \frac{Mv}{(m+M)} \quad \Rightarrow \quad \frac{m}{M} v
\]

So the added kinetic energy of the Earth is:

\[
\frac{1}{2} Mv^2 = \frac{1}{2} M \left( \frac{m}{M} \right)^2 v = \frac{1}{2} m v^2 \left( \frac{m}{M} \right) \quad \ll \quad \frac{1}{2} m v^2
\]

So the KE gained by the Earth is vastly smaller than the KE lost by the Snowball. You can make the same argument for the rotational energy of the Earth. So essentially none of the Snowball's KE goes into the Earth. Instead it goes into the form of heat which melts some of the snow and/or heats up the wall.

This is a general result for a small object hitting a large one. The large object picks up essentially all of the momentum, but essentially none of the energy (except possibly in the form of heat).
#2) Up, up and away:

a) The escape velocity can be found from energy conservation arguments alone. The key here is to recognize that the gravitational potential energy is not general, high, but rather is given by the general equation

$$ U(r) = -\frac{GMm}{r} $$

where

- $M =$ the mass of the planet
- $r =$ the distance from the centre of mass of the planet to the mass
- $m =$ the particle mass
- $G =$ Newton’s Constant $(6.67 \times 10^{-11} \text{ N m}^2 \text{kg}^{-2})$.

Note that at the surface of the Earth, $U(R) = -GMM/R.$

and at $r \to \infty$, $U(r) \to 0$ (which is the maximum value of $U(r)$)

Another key is to recognize that for the escape velocity, the particle will escape to $r=\infty$ and its velocity will approach zero.

So energy conservation tells us that, at launch,

$$ E_{\text{launch}} = \frac{1}{2} mv^2 - \frac{GMM}{r} $$

and at $r=\infty$

$$ E_{\infty} = 0 + 0 = 0. $$

$$ \therefore \frac{1}{2} mV_{\text{esc}}^2 = \frac{GMM}{R} $$

$$ V_{\text{esc}} = \sqrt{\frac{2GM}{R}} $$

Note that, as expected, $V_{\text{esc}}$ does not depend on the projectile mass!

For Earth, $M_\oplus = 6 \times 10^{24} \text{kg}$, $R = 6,400 \text{km}$ so $V_{\text{esc}} = \sqrt{\frac{2 \cdot (6.67 \times 10^{-11} \text{ N m}^2 \text{kg}^{-2}) (6 \times 10^{24} \text{kg})}{6.4 \times 10^6 \text{m}}} = 11.2 \text{km/s}$
#2)

b) \( V_{\text{esc}} = \sqrt{\frac{2GM}{R}} \)

For a constant density sphere, \( M = \frac{4}{3} \pi R^3 \rho \)

where \( \rho \) is the density

\[
V_{\text{esc}} = R \sqrt{\frac{8 \pi G \rho}{3}}
\]

\[
R = \sqrt{\frac{3 V_{\text{esc}}^2}{8 \pi G \rho}}
\]

To estimate a person's jumping speed, assume that a person can jump 1 meter high.

Then \( V_p^2 = V_o^2 - 2gd \)

\( v_o = V_o - 2gd \) \( \Rightarrow \)

\[
V_o^2 = (2 \times 10^{-10} \text{ m/s})(1 \text{ m})
\]

\[
V_o^2 = 2 \text{ m}^2/\text{s}^2
\]

Alternatively, from cons. of E:

\[
\frac{1}{2} m v^2 = m g h
\]

\[
v^2 = 2 g h
\]

The density of the Earth can be approximated as \( \rho = 5.5 \times 10^3 \text{ kg/m}^3 \)

which is \( \approx 5.5 \times \) more dense than water.

So, given a planet of radius \( R \) and density \( \rho \), you could jump with an escape velocity and leave the planet. That radius is:

\[
R = \sqrt{\frac{(3)(20 \text{ m}^2/\text{s}^2)}{8 \pi (6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(5.5 \times 10^3 \text{ kg/m}^3)}}
\]

\[
R = 2551 \text{ m}
\]

\[
R \approx 2.5 \text{ km}
\]

So be careful if you ever find yourself on a tiny planet with density comparable to that of earth!
#3) Leaving the Sphere

Start with Newton's Second Law. Since the particle moves with fixed radius but changing angle, let's use polar coordinates:

\[
\alpha_r = \dot{r} - r \dot{\theta}^2 = -r \ddot{\theta} = -\frac{v^2}{R}
\]

\[
F_r = F_N - mg \cos \theta
\]

\[
\alpha_r = \frac{1}{m} F_r
\]

\[
-\frac{v^2}{R} = \frac{1}{m} F_N - g \cos \theta
\]

The condition for losing contact is that \( F_N \to 0 \). So one constraint equation is

\[
v^2 = Rg \cos \theta \quad (*) \quad \text{(From Newton II and } F_N = 0).\]

Next we look at energy conservation to express \( v \) as a function of \( R \) and \( \theta \):

\[
\dot{E} = 0 \quad E_{\text{initial}} = mg \Delta R \quad \text{(particle sits at rest atop the sphere)}
\]

At some later time: \( E = \frac{1}{2} mv^2 + mgh \) where \( h = R + R \cos \theta = R(1 + \cos \theta) \)

so \( mg \Delta R = \frac{1}{2} mv^2 + mgR(1 + \cos \theta) \)

\[
\Rightarrow \quad v^2 = 2gR (1 - \cos \theta) \quad (**)\]

By equating \((*)\) and \((***)\) we find: \( \cos \theta = \frac{2}{3} \quad \Rightarrow \quad \theta = 48.2^\circ\).
#4) Roller Coaster

At the start (release) the ball's energy is $mg(2R+h)$. At the peak of the circular loop, the energy is $\frac{1}{2}mv^2 + mg2R$.

So: $mg(2R+h) = \frac{1}{2}mv^2 + mg2R$

$$v^2 = 2gh \quad (*)$$

We must then turn to Newton's Second Law to find out how large the velocity must be in order for the coaster to maintain contact with the track:

Given circular motion, we know:

$$a_r = \ddot{r} - r\ddot{\phi}^2 = -r\ddot{\phi}^2 = -\frac{v^2}{R}$$

And both $F_N$ and $F_g$ contribute to $a_r$:

$$a_r = \frac{1}{m}F_r + \frac{v^2}{R} = \frac{1}{m}(F_N + F_g) = \frac{1}{m}(F_N + mg) = \frac{1}{m}F_N + g$$

When the cart loses contact, $F_N \rightarrow 0$. So we require that $F_N \geq 0$

Therefore $$v^2 \geq Rg \quad (**)$$

Combining (*) with (**) we find a constraint on $h$:

$$2gh \geq Rg \quad \Rightarrow \quad h \geq \frac{1}{2}R$$

So you must release the coaster from at least $R/2$ above the top of the loop's peak.
#5) Down, then around on a cone:

When the ball is moving in a horizontal circle, there is no acceleration in the vertical direction, so the weight of the ball is countered by the vertical component of the normal force. If the cone's apex angle is \( \theta \), then the vertical component of \( \vec{N} \) is \( N \sin \theta \) and so:

\[
\begin{align*}
\frac{mg}{\sin \theta} &= N \\
mg &= N \sin \theta \\
N &= \frac{mg}{\sin \theta}
\end{align*}
\]

The horizontal component of \( \vec{N} \) provides the only radial force:

Newton's 2nd Law in Polar coordinates, radial component

\[
\alpha_r = \ddot{r} - r \dot{\phi}^2 = \frac{1}{m} F_r
\]

since \( r = r = 0 \) then \( -r \dot{\phi}^2 = -\frac{1}{m} N \cos \theta \) \( (*) \)

Now, \( r = h \tan \theta \) where \( h \) is the height of the horizontal circle above the cone's apex. And for circular motion, \( \phi' = \sqrt{r} \) so \( (*) \) becomes:

\[
\frac{-v^2}{htan \theta} = -\frac{1}{m} \frac{mg \cos \theta}{\sin \theta}
\]

\[
\Rightarrow \quad v^2 = gh
\]

So the ball must be moving with velocity \( \sqrt{gh} \) in order to move in a horizontal circle. Conservation of energy, then, will tell us the height.
#5) (cont.)

through which the ball must fall in order to acquire that velocity by the time it strikes the platform.

Let \( H \) be the initial height of the ball (where it is released from rest). Then the amount of gravitational potential energy that it loses is:

\[
mg(H-h)
\]

and this goes entirely into kinetic energy:

\[
\frac{1}{2}mv^2 = mg(H-h)
\]

and we know that \( v^2 = gh \) so

\[
\frac{1}{2}mg h = mg(H-h)
\]

\[
\frac{1}{2}h = H-h
\]

\[
\frac{3}{2}h = H
\]

\[
\therefore \quad \frac{H}{h} = \frac{3}{2}
\]

Note that this solution is independent of the absolute value of \( h \). You only need to know the ratio of \( H \) and \( h \) in order to determine the release height of the ball.