Mathematical Methods and Numerical Techniques I — Homework Problems

1. Some Series Involving $\pi$ (15 P.)
   (a) In Problem 1, Exercise Sheet 5, you found the series representation for $\arctan x$ around $x = 0$. (5 P.)
   Use it to verify the following statements:
   \[
   1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4},
   \]
   \[
   1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \ldots = \frac{\pi}{2\sqrt{3}}.
   \]
   (2)
   (b) Why is the series (2) useful in numerically computing $\pi$, but not the series (1)? Write a C++ (10 P.)
   program that calculates $\pi/(2\sqrt{3})$ to 10 decimals.

2. Complex Numbers (20 P.)
   (a) (Boas 2.4) Write in polar form $z = |z|e^{i\phi}$, and locate in the complex plane: (5 P.)
   \[
   z = i - 1, \quad z = 1 - i\sqrt{3}, \quad z = 2 - 2i, \quad z = 2(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}).
   \]
   (3)
   (b) (Boas 2.4) Write in cartesian form $z = x + iy$: (5 P.)
   \[
   z = \sqrt{2}e^{-i\pi/4}, \quad z = 4e^{2i\pi/3}, \quad z = e^{7i\pi}.
   \]
   (4)
   (c) (Boas 2.5) Simplify to cartesian form $z = x + iy$: (5 P.)
   \[
   z = \frac{1}{1 + i}, \quad z = (i + \sqrt{3})^2, \quad z = \left(\frac{1 + i}{1 - i}\right)^2, \quad z = \frac{3i - 7}{4 + i}.
   \]
   (5)
   (d) (Boas 2.9) Evaluate using Euler’s formula: (5 P.)
   \[
   z = (1 + i\sqrt{3})^6, \quad z = (1 - i)^8, \quad z = \left(\frac{i\sqrt{2}}{1 + i}\right)^8.
   \]
   (6)

3. Complex Transcendental Functions (15 P.)
   (a) (Boas 2.10) Find all roots of: (5 P.)
   \[
   z^3 = 27, \quad z^3 = -1, \quad z^5 = i.
   \]
   (7)
   (b) (Boas 2.10.28) Using Euler’s formula, show that: (5 P.)
   \[
   \cos 4\phi = \cos^4 \phi - 6 \sin^2 \phi \cos^2 \phi + \sin^4 \phi, \quad \sin 4\phi = 4 \sin \phi \cos^3 \phi - 4 \sin^3 \phi \cos \phi.
   \]
   (8)
   (c) (Boas 2.10.31) Prove that the sum of all $k$th roots of a complex number $z$ ($k = 2, 3, \ldots$) is always zero. (5 P.)
4. Hyperbolic Functions

(a) (Boas 2.12.1) Verify that with \( z = x + iy \),

\[
\sin z = \sin x \cosh y + i \cos x \sinh y .
\]  

(b) (Boas 2.12.8, 2.12.21) Show that for all complex \( z \),

\[
\cosh^2 z + \sinh^2 z = \cosh 2z , \quad \cosh^2 z - \sinh^2 z = 1 .
\]

(c) (Boas 2.12.19) Verify the addition theorem for \( \tanh x \):

\[
\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} .
\]

(d) Show (using part (b)) that:

\[
\cosh z = \frac{1}{\sqrt{1 - \tanh^2 z}} , \quad \sinh z = \frac{\tanh z}{\sqrt{1 - \tanh^2 z}} .
\]