Mathematical Methods and Numerical Techniques I — Homework Problems

1. **Reading Assignment**
   Read Boas, Chapter 3.10–3.11.

2. **Inverse Matrices**
   *(Boas 3.6.15)* Show that the following matrices $A, B, C$ are all regular, and find the respective inverse matrices $A^{-1}, B^{-1}, C^{-1}$:

   \[
   A = \begin{pmatrix}
   \cos \theta & \sin \theta \\
   -\sin \theta & \cos \theta
   \end{pmatrix}, \quad
   B = \begin{pmatrix}
   2 & 0 & -4 \\
   -1 & 1 & 1
   \end{pmatrix}, \quad
   C = \begin{pmatrix}
   0 & 1 & 0 & 0 \\
   1 & 0 & 1 & 0 \\
   0 & 1 & 0 & 1 \\
   0 & 0 & 1 & 0
   \end{pmatrix}.
   \]

3. **Determinants**
   *(Boas 3.3.1, 3.3.6)* Find the determinants $\det A, \det B$ of the following matrices:

   \[
   A = \begin{pmatrix}
   -2 & 3 & 4 \\
   3 & 4 & -2 \\
   5 & 6 & 3
   \end{pmatrix}, \quad
   B = \begin{pmatrix}
   0 & 1 & 1 & 1 \\
   1 & 0 & 1 & 1 \\
   1 & 1 & 0 & 1 \\
   1 & 1 & 1 & 0
   \end{pmatrix}.
   \]

4. **Eigenvalue Problems**
   Consider the following three matrices $A, B, C$:

   \[
   A = \begin{pmatrix}
   \cosh \chi & \sinh \chi \\
   \sinh \chi & \cosh \chi
   \end{pmatrix}, \quad
   B = \begin{pmatrix}
   1 & 3 \\
   2 & 2
   \end{pmatrix}, \quad
   C = \begin{pmatrix}
   1 & -1 & 0 \\
   0 & 1 & -1 \\
   -1 & 0 & 1
   \end{pmatrix}.
   \]

   (a) Show that the eigenvalues of $A$ are given by $\lambda_{\pm} = e^{\pm \chi}$. What are the corresponding (right) eigenvectors $u_{\pm}$? \hspace{1cm} (7 P.)

   (b) Find the eigenvalues $\lambda_1, \lambda_2$ of $B$, and establish a set of right eigenvectors $u_{1,2}$ with $Bu_{1,2} = \lambda_{1,2} u_{1,2}$, and a set of corresponding left eigenvectors $V_{1,2}$ with $v_{1,2}^T B = \lambda_{1,2} v_{1,2}^T$. Verify that $v_1 \cdot u_2 = u_1 \cdot v_2 = 0$. \hspace{1cm} (7 P.)

   (c) Show that the matrix $C$ is normal, i.e., $C^T C = CC^T$ holds. \hspace{1cm} (3 P.)

   (d) Verify that the three eigenvalues of $C$ are:

   \[
   \lambda_0 = 0, \quad \lambda_{\pm} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i.
   \]

   \[\text{— 1 —}\]
(e) Find the corresponding (right) eigenvectors of $C$, and show that a possible set is given by:

$$
u_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \nu_+ = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}, \quad \nu_- = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$ is one of the third roots of unity. Verify that the left eigenvectors $v_0, v_\pm$ are the complex conjugates of the right eigenvectors (apart from scaling):

$$v_0 = u_0, \quad v_+ = u_-^*, \quad v_- = u_+^* = u_+.$$