1. Gaussian, or normal distribution. A *Gaussian* probability distribution, also known as a normal distribution or bell curve, is

\[ P(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u - u_0)^2}{2\sigma^2}}, \]

where \( u \) is a random variable, and \( \sigma \) and \( u_0 \) are constants.

(a) On symmetry grounds, the expectation of \( u \) is

\[ \langle u \rangle = u_0, \]

since the probability distribution is symmetric about \( u = u_0 \). In statistics, the *variance* of \( u \) is the expectation value of \((u - \langle u \rangle)^2\):

\[ \langle (u - u_0)^2 \rangle = \int_{-\infty}^{\infty} P(u)(u - u_0)^2 du. \]

You are given that \( P(u) \) is normalized, \( \int_{-\infty}^{\infty} e^{-(u-u_0)^2/(2\sigma^2)} du = \sqrt{2\pi} \sigma \). Using the trick of differentiating both sides of this equation with respect to \( \sigma \), show that the variance is \( \langle (u - u_0)^2 \rangle = \sigma^2 \). The square root of variance is known as the *standard deviation*. Therefore, \( \sigma \) is the standard deviation of a Gaussian distribution.
2. **Position and momentum uncertainty of a Gaussian wavepacket.** Before stating the problem, here is some background for orientation. In class, we obtained the following relations between the position space and momentum space wavefunctions $\Psi(x, t)$ and $\tilde{\Psi}(p, t)$:

$$
\Psi(x, t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\Psi}(p, t) e^{i\frac{p}{\hbar}x},
$$

$$
\tilde{\Psi}(p, t) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \Psi(x, t) e^{-i\frac{p}{\hbar}x},
$$

where $\tilde{\Psi}(p, t) = \Psi(p, 0)e^{-i\omega t}$. We will focus on the first relation in this problem. Let lowercase $\tilde{\psi}(p)$ denote $\Psi(p, 0)$. Since $E = \hbar\omega$, we have

$$
\tilde{\Psi}(p, t) = \tilde{\psi}(p)e^{-i\frac{Et}{\hbar}},
$$

and Eq. (4) becomes

$$
\Psi(x, t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)e^{i\frac{p}{\hbar}(px - Et)}.
$$

Provided that $E(p) = \frac{p^2}{2m}$, this wavefunction solves the free particle Schrödinger equation. Now, here’s the problem:

(a) Suppose that the momentum space wavefunction at time $t = 0$ is

$$
\tilde{\psi}(p) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}(p-p_0)^2},
$$

for some real parameter $\alpha$. Then, the probability density for observing the particle with momentum $p$ at time $t$,

$$
\tilde{P}(p) = |\tilde{\Psi}(p, t)|^2 = |\tilde{\psi}(p)|^2,
$$

is a Gaussian distribution, of the form studied in Problem 1. Using Problem 1, please determine the expectation value $\langle p \rangle$ and the uncertainty $\Delta p$ in terms of $\alpha$.

(b) One of the neat things about Gaussian distributions is that a Gaussian in momentum space leads to a Gaussian in position space. For the choice (8), the integral in Eq. (7) can be performed explicitly. The resulting wavepacket can be shown to satisfy

$$
|\Psi(x, t)|^2 = \frac{1}{\sqrt{\pi \alpha \hbar^2 \left(1 + \left(\frac{t}{m\alpha\hbar}\right)^2\right)}} \exp\left(\frac{-\left(x - \frac{p_0}{m}\right)^2}{\alpha \hbar^2 \left(1 + \left(\frac{t}{m\alpha\hbar}\right)^2\right)}\right),
$$

Using this result, together with Problem 1, please determine the expectation value $\langle x \rangle$ and the uncertainty $\Delta x$ at time $t$, in terms of $\alpha$. With what velocity does the wavepacket
propagate? Confirm that this is the same as the group velocity \( v_g = \frac{d\omega}{dk} \) at \( k_0 = p_0/\hbar \).

Does the wavepacket retain its shape as time evolves?

(c) Finally, compute the product \( \Delta x \Delta p \), and show that it satisfies Heisenberg’s uncertainty principle

\[
\Delta x \Delta p \geq \frac{\hbar}{2}.
\]

with equality at time \( t = 0 \). In fact, a Gaussian wavepacket is the unique in this regard. It is the only wavefunction that attains the lower bound in the uncertainty product.

3. Probability density for a Gaussian wavepacket. In this problem we will derive the probability density (10) of a Gaussian wavepacket. For the momentum space wavefunction (8), the integral (7) becomes

\[
\Psi(x, t) = \frac{1}{\sqrt{2\pi \hbar}} \left( \frac{\alpha}{\pi} \right)^{1/4} \int_{-\infty}^{\infty} dp e^{-\frac{\alpha}{2}(p-p_0)^2 + \frac{i}{\hbar}(px - \frac{p^2t}{2m})}.
\]

(a) By expanding the exponent, grouping like powers of \( p \), and then “completing the square,” i.e., writing

\[
-\alpha p^2 + bp = -a \left( p - \frac{b}{2a} \right)^2 + \frac{b^2}{4a},
\]

show that the exponent in the integral becomes

\[
-\gamma^2 \left[ p - \frac{(\alpha p_0 + i\hbar/2\gamma^2)^2}{4\gamma^2} \right]^2 + \frac{(\alpha p_0 + i\hbar/2\gamma^2)^2}{4\gamma^2} - \alpha p_0^2.
\]

where \( \gamma^2 = \frac{\alpha}{2} + \frac{\hbar^2}{2m\hbar} \). Therefore, the wavefunction becomes

\[
\Psi(x, t) = \frac{1}{\sqrt{2\pi \hbar}} \left( \frac{\alpha}{\pi} \right)^{1/4} e^{-\frac{\alpha}{2}(p_0^2)} \int_{-\infty}^{\infty} dp e^{-\gamma^2 \left[ p - \frac{(\alpha p_0 + i\hbar/2\gamma^2)^2}{4\gamma^2} \right]^2}.
\]

(b) The integral on the right hand side can be evaluated using \( \int_{-\infty}^{\infty} e^{-(u-u_0)^2/(2\sigma^2)} du = \sqrt{2\pi} \sigma \) from Problem 1. Show that this leads to

\[
|\Psi(x, t)|^2 = \frac{1}{2\hbar} \sqrt{\frac{\alpha}{\pi} \frac{1}{|\gamma|^2}} e^{-\alpha p_0^2} \left| e^{\left( \frac{(\alpha p_0 + i\hbar/2\gamma^2)^2}{4\gamma^2} \right)} \right|^2.
\]

(c) Finally, the exponential factors on the right hand side are

\[
\exp \left[ -\alpha p_0^2 \right] \cdot \exp \left[ \left( \frac{\alpha p_0 - i\hbar/\gamma^2}{2\gamma^2} \right)^2 + \left( \frac{\alpha p_0 + i\hbar/\gamma^2}{2\gamma^2} \right)^2 \right] = \exp \left[ -\frac{\alpha(x - \frac{p_0}{m}t)^2}{4\hbar^2|\gamma|^4} \right],
\]

where I’ve spared you some messy algebra in the last step. (Here, \( \exp u = e^u \).)
By evaluating $|\gamma|^2$, show that

$$|\Psi(x, t)|^2 = \frac{1}{\sqrt{\pi \alpha \hbar^2 \left(1 + \left(\frac{t}{m \alpha \hbar}\right)^2\right)}} \exp\left(-\frac{(x - \frac{p_0 t}{m})^2}{\alpha \hbar^2 \left(1 + \left(\frac{t}{m \alpha \hbar}\right)^2\right)}\right),$$

as claimed in Problem 2.

4. **Probability density and current for the Klein-Gordon equation.** For the 1D Klein-Gordon equation,

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \Psi = 0,$$  \hfill (18)

we define the probability current to be

$$j_x = \text{Re} \left[ \Psi^* \frac{\hat{p}}{m} \Psi \right] = -\frac{i \hbar}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right),$$  \hfill (19)

just as we did for the Schrödinger equation. On the other hand, the probability density is different.

(a) For the Klein-Gordon equation, show that the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} = 0$$  \hfill (20)

is satisfied, provided we define the probability density to be

$$\rho = \frac{i \hbar}{2mc^2} \left( \Psi^* \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t} \Psi \right),$$  \hfill (21)

which differs from the $|\Psi|^2$ probability density of the Schrödinger equation.

(b) Consider the wavefunction $\Psi(x, t) = Ne^{i(kx - \omega t)}$. Find the dispersion relation $\omega = \omega(k)$ such that this wavefunction satisfies the Klein-Gordon equation. Show that it is the same as the relation obtained from the relativistic energy-momentum equation $E^2 = p^2 c^2 + m^2 c^4$ and the quantum mechanical relations $E = \hbar \omega$, $p = \hbar k$.

(c) For the same wavefunction and dispersion relation $\omega(k)$, compute $\rho$ and $j_x$.

(d) Show that $j_x = \rho v$. (In class, we will see that a similar relation holds for the Schrödinger equation, and for charge and current density in electromagnetism.) Identify the coefficient as $v$ in two ways: (i) Show that it is the group velocity. (2) Show that it is $pc^2/E$; then, from the relativistic expressions for $p(v)$ and $E(v)$ for a point particle of mass $m$ (reviewed in recitation section Friday 2/28), show that this ratio is again $v$. 


Feedback. There will not be graded feedback this week. Instead, please take the time to complete a midsemester evaluation form, to offer your thoughts on how the class is going, and how it can be improved.