Problem Set 7

Due: Thu 28 March 2013

Reading:

Thu 3/21: Townsend Sec. 4.1.

Fri 3/22: Townsend Sec. 4.2, French & Taylor Handout.

Thu 3/28: Townsend Secs. 4.2, French & Taylor Handout.

Fri 3/29: Townsend Secs. 4.3.

Reminder: Exam 2 will be distributed on Friday 29 March. It is a closed book 2.5 hour take-home exam, but you may prepare one page of notes (front and back) to use during the exam. It will be due on Friday 5 April in class. The exam will cover the material in the textbook through Sec. 4.1 and in the homework through Problem Set 7.

Problems: Please choose any five of the following six problems.

1. Gaussian, or normal distribution. A Gaussian probability distribution, also known as a normal distribution or bell curve, is

\[ P(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-u_0)^2}{2\sigma^2}}, \]

where \( u \) is a random variable, and \( \sigma \) and \( u_0 \) are constants.

(a) On symmetry grounds, the expectation of \( u \) is

\[ \langle u \rangle = u_0, \]

since the probability distribution is symmetric about \( u = u_0 \). In statistics, the variance of \( u \) is the expectation value of \( (u - \langle u \rangle)^2 \):

\[ \langle (u - u_0)^2 \rangle = \int_{-\infty}^{\infty} P(u)(u-u_0)^2 du. \]
You are given that $P(u)$ is normalized, $\int_{-\infty}^{\infty} e^{-(u-u_0)^2/(2\sigma^2)} du = \sqrt{2\pi} \sigma$. Using the trick of differentiating both sides of this equation with respect to $\sigma$, show that the variance is $\langle (u-u_0)^2 \rangle = \sigma^2$. The square root of variance is known as the standard deviation. Therefore, $\sigma$ is the standard deviation of a Gaussian distribution.

2. Position and momentum uncertainty of a Gaussian wavepacket. Before stating the problem, here is some background for orientation. In class, we obtained the following relations between the position space and momentum space wavefunctions $\Psi(x,t)$ and $\tilde{\Psi}(p,t)$:

\begin{align}
\Psi(x,t) &= \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\Psi}(p,t)e^{i\frac{p}{\hbar}x}, \\
\tilde{\Psi}(p,t) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \Psi(x,t)e^{-i\frac{p}{\hbar}x},
\end{align}

where $\tilde{\Psi}(p,t) = \Psi(p,0)e^{-i\omega t}$. We will focus on the first relation in this problem. Let lowercase $\tilde{\psi}(p)$ denote $\Psi(p,0)$. Since $E = \hbar \omega$, we have

$$\tilde{\Psi}(p,t) = \tilde{\psi}(p)e^{-i\frac{Et}{\hbar}},$$

and Eq. (4) becomes

$$\Psi(x,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)e^{i\frac{p}{\hbar}(px-\omega t)}.$$

Provided that $E(p) = \frac{p^2}{2m}$, this wavefunction solves the free particle Schrödinger equation. Now, here’s the problem:

(a) Suppose that the momentum space wavefunction at time $t = 0$ is

$$\tilde{\psi}(p) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}(p-p_0)^2},$$

for some real parameter $\alpha$. Then, the probability density for observing the particle with momentum $p$ at time $t$,

$$\tilde{P}(p) = |\tilde{\Psi}(p,t)|^2 = |\tilde{\psi}(p)|^2,$$

is a Gaussian distribution, of the form studied in Problem 1. Using Problem 1, please determine the expectation value $\langle p \rangle$ and the uncertainty $\Delta p$ in terms of $\alpha$.

(b) One of the neat things about Gaussian distributions is that a Gaussian in momentum space leads to a Gaussian in position space. For the choice (8), the integral in Eq. (7) can be performed explicitly. The resulting wavepacket can be shown to satisfy

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi\alpha\hbar^2\left(1 + \left(\frac{t}{\alpha\hbar}\right)^2\right)^2}} \exp\left(\frac{-(x-p_0t)^2}{\alpha\hbar^2\left(1 + \left(\frac{t}{\alpha\hbar}\right)^2\right)}\right).$$
Using this result, together with Problem 1, please determine the expectation value $\langle x \rangle$ and the uncertainty $\Delta x$ at time $t$, in terms of $\alpha$. With what velocity does the wavepacket propagate? Confirm that this is the same as the group velocity $v_g = d\omega/dk$ at $k_0 = p_0/\hbar$. Does the wavepacket retain its shape as time evolves?

(c) Finally, compute the product $\Delta x \Delta p$, and show that it satisfies Heisenberg’s uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

with equality at time $t = 0$. In fact, a Gaussian wavepacket is the unique in this regard. It is the only wavefunction that attains the lower bound in the uncertainty product.


In part (a), please provide an explanation for the given equation,

$$\Delta v = \frac{\Delta p}{2m} \sim \frac{\hbar}{2m \Delta x}.$$

After doing parts (a) and (b), please add an additional part (c):

(c) In Problem 2 of Problem Set 6, we studied a Gaussian wavepacket, parametrized by two numbers $\alpha$ and $p_0$. We found that, at time $t$,

$$\langle p \rangle = p_0, \quad \Delta p = \frac{1}{\sqrt{2\alpha}},$$

$$\langle x \rangle = \frac{p_0}{m} t, \quad \Delta x = \sqrt{\frac{\alpha \hbar^2}{2}} \sqrt{1 + \left(\frac{t}{m \alpha \hbar}\right)^2}.$$

Please rewrite the equation for $\Delta x$, expressing the right hand side in terms of $\hbar$, $t$, $m$, and $\Delta x_0 = \sqrt{\alpha \hbar^2 / 2}$ only, where $\Delta x_0$ is the uncertainty $\Delta x$ at time $t = 0$. Compute the exact time $t$ such that the width of the wavepacket is twice the width at time $t = 0$ (i.e., such that $\Delta x = 2\Delta x_0$), and show that it agrees with your estimate in part (a) up to a numerical factor.

4. Probability conservation on the half-line.

Consider a quantum mechanical free particle ($V(x) = 0$) living on a one dimensional half-line $x \geq 0$, and described by the Schrödinger equation.

(a) Assuming that $\Psi(x, t) \to 0$ as $x \to \infty$, what is the condition on $\Psi(x, t)$ at $x = 0$, so that no probability “leaks out” at the origin, that is, so that the total probability is conserved?
(b) The condition found in part (a) can be enforced with the appropriate choice of boundary condition at the origin. One choice would be $\Psi(0, t) = 0$, which physically corresponds to choosing infinite potential at $x = 0$. Consider the more general boundary condition

$$
\frac{\partial \Psi}{\partial x}(0, t) = \mu \Psi(0, t), \quad \frac{\partial \Psi^*}{\partial x}(0, t) = \mu^* \Psi^*(0, t),
$$

for some complex number $\mu$. Please write down an expression for the probability current $j_x(0, t)$ at $x = 0$, using Eq. (1) to eliminate derivatives of $\Psi$. What is the condition on $\mu$ so that probability is conserved? (You should find that an arbitrary complex number will not lead to probability conservation at $x = 0$.)

(c) Suppose that there exists a normalizable stationary state of negative energy. What further condition must be imposed on $\mu$, and what is the explicit wavefunction $\Psi(x, t)$?

5. **Probability of observing energy $E_n$ in a non-stationary state.** Townsend 3.8. We will need the integrals

$$
\int y \sin y \, dy = -y \cos y + \sin y,
\int y^2 \sin y \, dy = -y^2 \cos y + 2y \sin y + 2 \cos y.
$$

Please derive these results using integration by parts.

6. **Energy eigenfunctions, eigenvalues, and probability.** Townsend 3.10. We will need the integral

$$
\int y \cos y \, dy = y \sin y + \cos y.
$$

Please derive this result using integration by parts.

7. **Feedback.** By Thursday of each week, please send me an email message to provide feedback on the class and on your reading. (My email address is mbschulz at brynmawr.edu). For example: Which parts were easier or harder to understand? Do you have any questions that you would like to clarify or areas where you would like more practice in recitation section? Was there something that you found particularly interesting or uninteresting? Was the problem set of reasonable length and difficulty. If you have any thoughts on how to improve the textbook for future students using future editions, please let me know and I will pass that information on to the author, John Townsend. The purpose of the feedback is to help you to reflect on your learning process and to provide me with brief but valuable information that will help to make this class the best possible experience for everyone.