Probability Conservation

**Probability density (new notation \( \rho \) "rho"):**

\[ \rho(x,t) = \psi^\ast(x,t) \psi(x,t) \]

Rate of change in probability density:

\[ \frac{\partial \rho}{\partial t} = \psi^\ast \frac{\partial \psi}{\partial t} + \frac{\partial \psi^\ast}{\partial t} \psi \]  \( \quad (1) \)

Use Schrödinger equation to re-express right hand side in terms of \( x \) derivatives:

\[ \frac{\partial \psi}{\partial t} = \pm \frac{\hbar}{2m} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi \right) \]  \( \text{(Schrödinger Eq.)} \),

\[ \frac{\partial \psi^\ast}{\partial t} = \pm \frac{\hbar}{2m} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^\ast}{\partial x^2} + V(x) \psi^\ast \right) \]  \( \text{(complex conjugate)} \).

Substitute into Eq. (1). \( \psi^\ast V(x) \psi \) terms cancel, leaving

\[ \frac{\partial \rho}{\partial t} = -\frac{\hbar}{2im} \left[ \psi^\ast \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^\ast}{\partial x^2} \psi \right] \]

\[ = -\frac{\hbar}{2im} \left[ \psi^\ast \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi^\ast}{\partial x^2} \right] \]

\[ = -\frac{\hbar}{2im} \left[ \psi^\ast \frac{\partial \psi}{\partial x} - \frac{\partial \psi^\ast}{\partial x} \psi \right] \]

\[ \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} = 0 \]  \( \text{ Conservation of probability eq. } \)

(differential form), where the probability current \( j_x(x,t) \) is

\[ j_x = \frac{\hbar}{2im} \left[ \psi^\ast \frac{\partial \psi}{\partial x} - \frac{\partial \psi^\ast}{\partial x} \psi \right] \]

\[ = \frac{\hbar}{2} \left[ \psi^\ast \left( -i \hbar \frac{\partial}{\partial x} \right) \psi + \text{complex conjugate} \right] \]

\[ = \frac{\hbar}{2} \left[ \psi^\ast \left( \frac{\hat{p}}{\hbar} \right) \psi \right], \quad \text{Re} \left[ \right] = \text{"real part"}. \]

(Do not confuse \( \rho \) (prob density) and \( \hat{p} \) (momentum).)
What is the rate of change in the probability of observing the particle in the interval $a \leq x \leq b$? 

\[ P(a \leq x \leq b, t) = \int_a^b dx \mathcal{E}_{*}^\dagger \mathcal{E} = \int_a^b dx \rho \]

\[ \frac{d}{dt} P(a \leq x \leq b, t) = \frac{d}{dt} \int_a^b dx \rho(x, t) = \int_a^b dx \frac{\partial \rho}{\partial t}(x, t) \]

Conservation of probability eq. (integral form): (*)

\[ = - \int_a^b dx \frac{\partial \rho}{\partial x}(x, t) \]

\[ = - j_x(b, t) + j_x(a, t). \]  

Why out of/into? Sign conventions:

- $j_x > 0$: probability flows to right  \text{(positive x-direction)}
- $j_x < 0$: probability flows to left \text{(negative x-direction)}

Conservation of total probability? \((a = -\infty, b = +\infty)\)

Guaranteed if \(j(\pm \infty, t) = 0\) \text{(no probability current flowing in or out at } x = \pm \infty).\)

We usually assume that \(\mathcal{E}(x, t) \to 0\) as \(x \to \pm \infty\), since this is required in order to normalize the wavefunction \((\text{i.e., for } \int_{-\infty}^{\infty} dx |\mathcal{E}|^2 \text{ to converge}). \) So, indeed \(j_x(\pm \infty, t) = 0.\)

Comment: On Problem Set, consider a particle on the half-line \(0 \leq x \leq \infty\). In this case, don’t want any probability current flowing in or out at \(x = 0, \infty.\)