Problem: Probability current of a nonrelativistic free particle

(a) What is the dispersion relation of a nonrelativistic free particle in quantum mechanics? (b) Compute the group velocity and show that it is equal to the classical velocity of the particle. (c) For $\Phi(x,t) = N e^{i(kx-\omega t)}$, compute the probability density $P$ and probability current $j_x$. (d) Does your result for part (c) make sense?

Solution:

(a) "Free" means that $V(x) = 0$, so $E = \text{kinetic energy}$.

$$E = \frac{p^2}{2m} \quad \text{(classical K.E. $\frac{1}{2}mv^2$)}$$

Using $E = \hbar \omega$ and $p = \hbar k$, this becomes

$$\hbar \omega = \frac{\hbar^2 k^2}{2m} \implies \frac{\hbar k^2}{2m} \quad \text{(dispersion relation $\omega = \omega(k)$ for a nonrelativistic free particle)}$$

(b) $U_g = \frac{d\omega}{dk} = \frac{\hbar k}{m}$ (group velocity)

Or, in terms of $p$,

$$U_g = \frac{p}{m}$$

(c) $\Phi(x,t) = N e^{i(kx-\omega t)}$

Probability density: $P = \Phi^* \Phi = |\Phi|^2 = |N|^2$

Probability current:

$$j_x = \text{Re} \left[ \frac{\Phi^*}{m} \frac{\partial \Phi}{\partial t} \right] = -\frac{i\hbar}{2m} \left[ \Phi^* \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi^*}{\partial x} \Phi \right]$$
Using the first expression (with \( p = -i\hbar \frac{\partial}{\partial x} \)),

\[
    j_x = \text{Re} \left[ N^* e^{-i(kx - \omega t)} \left( -i\frac{\hbar}{m} \frac{\partial}{\partial x} \right) N e^{i(kx - \omega t)} \right] \\
    = \text{Re} \left[ N^* e^{-i(kx - \omega t)} \frac{\hbar k}{m} N e^{i(kx - \omega t)} \right] \\
    = \text{Re} \left[ \frac{\hbar k}{m} \right] \cdot \text{Im} \left[ \frac{\hbar k}{m} \right] = \text{Im} \left[ \frac{\hbar k}{m} \right]
\]

Since we found that \( p = \text{Im} \left[ \frac{\hbar k}{m} \right] \) and \( n_s = \frac{\hbar k}{m} \), we can write this as

\[
    j_x = p n_s .
\]

(d) Interpretation: similar to electromagnetism \( (e < 0 \text{ here}) \)

\[
\begin{align*}
\text{Change density} \quad & \vec{\rho}_{\text{elec}} = ne \\
\text{Current density} \quad & \vec{j}_{\text{elec}} = ne \vec{v}, \quad \vec{v} = \text{their velocity}
\end{align*}
\]

Consider \( N \) electrons in a volume \( V \) that strike an area \( A \) in time \( T \).

\[
N = nV \quad (\text{# electrons}) \\
L = vT \quad (\text{length of region that will strike } A \text{ in time } T)
\]

So, rate at which electrons strike the area \( A \):

\[
\left( \frac{\text{# electrons}}{\text{time}} \right) = \frac{nV}{T} = \frac{n(AL)}{T} = \frac{n(AvT)}{T} = nuA
\]

Rate at which charge strikes the area \( A \), i.e., current:

\[
I = \left( \frac{\text{change}}{\text{time}} \right) = \left( \frac{\text{#electrons}}{\text{time}} \right) e = n\nu eA .
\]
Electric current density

\[ j_{\text{elec}} = \frac{I}{A} = \left( \frac{\text{charge}}{\text{area} \cdot \text{time}} \right) = \frac{\text{neu}}{A} = \text{neu}^- . \]

I.e., \[ j_{\text{elec}} = P_{\text{elec}} \mathbf{v} \] in electromagnetism.

So, not surprising that

\[ j = \rho \mathbf{v} \]

for probability current and probability density too.

(If the analogy looks imperfect, it's because we're doing quantum mechanics in one dimension, so that \( \rho \) = probability per length (in x-direction) and \( j_x \) = probability current (in x-direction), with no transverse area.

In three dimensions, we'll see that \( \rho \) = probability per volume, and \( \mathbf{j} = \text{Re} \left[ \mathbf{\Pi} \frac{\mathbf{E}}{m} \right] = \text{probability current density} \) (i.e., per area), so the analogy works perfectly in this case, with probability in \( \rho \), \( \mathbf{j} \) of quantum mechanics playing the role of charge in \( P_{\text{elec}}, j_{\text{elec}} \) of electromagnetism.)