Physics 214: Introduction to Quantum Mechanics
Spring 2015

Problem Set 7a

Due: Fri 27 March 2015

Reading:

Make-up lecture: Townsend Sec. 4.2, French & Taylor Handout 1.

Mon 3/23: Townsend Sec. 4.3, French & Taylor Handout 2.

Wed 3/25: Townsend Secs. 4.4.

Fri 3/27: Townsend Secs. 4.6.

Reminder: Exam 2 will be distributed on Friday 27 March. It is a closed book 2.5 hour take-home exam, but you may prepare one page of notes (front and back) to use during the exam. It will be due on Wed 3 April in class. The exam will cover the material though this problem set, which corresponds to the textbook through the end of Chapter 3 plus the lecture notes on position and momentum space wavefunctions.

Problems:

1. A simple explanation for the spreading of wavepackets. Townsend 2.21.

In part (a), please provide an explanation for the given equation,

\[ \Delta v = \frac{\Delta p}{2m} \sim \frac{\hbar}{2m\Delta x}. \]

After doing parts (a) and (b), please add an additional part (c):

(c) In Problem 2 of Problem Set 7b, we will study the Gaussian wavepacket—a wavepacket whose position space and momentum space probability densities are both Bell curves. The momentum momentum space wavefunction at time \( t = 0 \) given by

\[ \tilde{\psi}(p) = \left( \frac{\alpha}{\pi} \right)^{1/4} e^{-\frac{\alpha}{2}(p-p_0)^2}, \]  

(1)
parametrized by two numbers $\alpha$ and $p_0$. We will find that, at time $t$,

$$\langle p \rangle = p_0, \quad \Delta p = \frac{1}{\sqrt{2\alpha}},$$

$$\langle x \rangle = \frac{p_0}{m} t, \quad \Delta x = \sqrt{\frac{\alpha \hbar^2}{2}} \sqrt{1 + \left( \frac{t}{m \alpha \hbar} \right)^2}.$$  

Please rewrite the equation for $\Delta x$, expressing the right hand side in terms of $\hbar$, $t$, $m$, and $\Delta x_0 = \sqrt{\alpha \hbar^2/2}$ only, where $\Delta x_0$ is the uncertainty $\Delta x$ at time $t = 0$. Compute the exact time $t$ such that the width of the wavepacket is twice the width at time $t = 0$ (i.e., such that $\Delta x = 2\Delta x_0$), and show that it agrees with your estimate in part (a) up to a numerical factor.

2. **Probability conservation on the half-line.** Consider a quantum mechanical free particle ($V(x) = 0$) living on a one dimensional half-line $x \geq 0$, and described by the Schrödinger equation.

(a) Assuming that $\Psi(x,t) \to 0$ as $x \to \infty$, what is the condition on $\Psi(x,t)$ at $x = 0$, so that no probability “leaks out” at the origin, that is, so that the total probability is conserved?

(b) The condition found in part (a) can be enforced with the appropriate choice of boundary condition at the origin. One choice would be $\Psi(0,t) = 0$, which physically corresponds to choosing infinite potential at $x = 0$. Consider the more general boundary condition

$$\frac{\partial \Psi}{\partial x}(0,t) = \mu \Psi(0,t), \quad \frac{\partial \Psi^*}{\partial x}(0,t) = \mu^* \Psi^*(0,t),$$

for some complex number $\mu$. Please write down an expression for the probability current $j_x(0,t)$ at $x = 0$, using Eq. (1) to eliminate derivatives of $\Psi$. What is the condition on $\mu$ so that probability is conserved? (You should find that an arbitrary complex number will not lead to probability conservation at $x = 0$.)

(c) Suppose that there exists a normalizable stationary state of negative energy. What further condition must be imposed on $\mu$, and what is the explicit wavefunction $\Psi(x,t)$?

3. **Probability of observing energy $E_n$ in a non-stationary state.** Townsend 3.8. We will need the integrals

$$\int y \sin y \, dy = -y \cos y + \sin y,$$

$$\int y^2 \sin y \, dy = -y^2 \cos y + 2y \sin y + 2 \cos y.$$

Please derive these results using integration by parts.
4. **Energy eigenfunctions, eigenvalues, and probability.** Townsend 3.10. We will need the integral
\[ \int y \cos y \, dy = y \sin y + \cos y. \]
Please derive this result using integration by parts.

5. **Feedback.** By Thursday of each week, please send me an email message to provide feedback on the class and on your reading. (My email address is mbschulz at brynmawr.edu). For example: Which parts were easier or harder to understand? Do you have any questions that you would like to clarify or areas where you would like more practice in recitation section? Was there something that you found particularly interesting or uninteresting? Was the problem set of reasonable length and difficulty. If you have any thoughts on how to improve the textbook for future students using future editions, please let me know and I will pass that information on to the author, John Townsend. The purpose of the feedback is to help you to reflect on your learning process and to provide me with brief but valuable information that will help to make this class the best possible experience for everyone.