Questions:

1. At time \( t = 0 \), the wavefunction for an electron in a one dimensional infinite square well \((V(x) = 0 \text{ for } 0 < x < L \text{ and } V(x) = \infty \text{ otherwise})\) is given by \( \Psi(x,0) = \sqrt{2/L} \sin(5\pi x/L) \).
Which one of the following is the true statement about the probability density \( |\Psi(x,t)|^2 \) at a later time \( t \)?

A. \( |\Psi(x,t)|^2 = \frac{2}{L} \sin^2(5\pi x/L) \cos^2(E_t/\hbar) \)
B. \( |\Psi(x,t)|^2 = \frac{2}{L} \sin^2(5\pi x/L) \exp(-i \cdot 2E_t/\hbar) \)
C. \( |\Psi(x,t)|^2 = \frac{2}{L} \sin^2(5\pi x/L) \sin^2(E_t/\hbar) \)
D. \( |\Psi(x,t)|^2 = \frac{2}{L} \sin^2(5\pi x/L), \) which is time-independent
E. None of the above

2. The wave function for an electron in an infinite square well at time \( t = 0 \) is given by \( \Psi(x,0) = A \sin^5(\pi x/L) \), where \( A \) is a suitable normalization constant. Which one of the following is the true statement of the probability density \( |\Psi(x,t)|^2 \) at a later time \( t \)?

A. \( |\Psi(x,t)|^2 = |A|^2 \sin^{10}(\pi x/L) \cos^2(E_t/\hbar) \)
B. \( |\Psi(x,t)|^2 = |A|^2 \sin^2(5\pi x/L) \exp(-i \cdot 2E_t/\hbar) \)
C. \( |\Psi(x,t)|^2 = |A|^2 \sin^2(5\pi x/L) \sin^2(E_t/\hbar) \)
D. \( |\Psi(x,t)|^2 = |A|^2 \sin^2(5\pi x/L), \) which is time-independent
E. None of the above

3. Energy expectation and uncertainty in our sloshing example. Calculate \( \langle E \rangle \), \( \langle E^2 \rangle \), and \( \Delta E \) for the wavefunction of our example of sloshing in the infinite square well,

\[ \Psi(x,t) = \frac{1}{\sqrt{2}} \Psi_1(x,t) + \frac{1}{\sqrt{2}} \Psi_2(x,t). \]
Example 3.3

Calculate $\langle E \rangle$, $\langle E^2 \rangle$, and $\Delta E$ for the wavefunction of our sloshing example

$$\Psi(x,t) = \frac{1}{\sqrt{2}} \psi_1(x,t) + \frac{1}{\sqrt{2}} \psi_2(x,t).$$

Solution

$$\Psi(x,t) = \sum_n c_n \psi_n(x), \quad \text{for} \quad c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{1}{\sqrt{2}}, \quad c_n = 0, \quad \text{otherwise}.$$

$$\langle E \rangle = \sum_n P_n E_n = \sum_n |c_n|^2 E_n = \frac{1}{2} E_1 + \frac{1}{2} E_2 = \frac{1}{2} (E_1 + E_2).$$

$$\langle E^2 \rangle = \sum_n P_n E_n^2 = \sum_n |c_n|^2 E_n^2 = \frac{1}{2} E_1^2 + \frac{1}{2} E_2^2 = \frac{1}{2} (E_1^2 + E_2^2).$$

$$\langle \Delta E \rangle^2 = \langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{4} (E_1^2 + E_2^2) - \left[ \frac{1}{2} (E_1 + E_2) \right]^2$$

$$= \frac{1}{4} (E_1^2 + E_2^2) - \frac{1}{4} (E_1^2 + E_2^2 + 2E_1E_2)$$

$$= \frac{1}{4} (E_1^2 + E_2^2 - 2E_1E_2)$$

$$= \left[ \frac{1}{2} (E_2 - E_1) \right]^2$$

$$\Delta E = \frac{1}{2} (E_2 - E_1)$$

(Note: When we studied the sloshing example, we wrote $\Delta E = E_2 - E_1$ (energy difference) to simplify our writing, but that wasn't the uncertainty in energy. As we now see, the energy uncertainty of the wavefunction $\Psi(x,t) = \frac{1}{\sqrt{2}} \psi_1(x,t) + \frac{1}{\sqrt{2}} \psi_2(x,t)$ is $\Delta E = \frac{1}{2} (E_2 - E_1)$. )
Problem
Show that the functions $\psi_n(x)$ for the particle in a box are orthonormal.

Solution
We wish to show that

$$\int_{-\infty}^{\infty} dx \, \psi_m^*(x) \psi_n(x) = \delta_{mn} = \sum_{n=0}^{\infty} \delta(n - m),$$

for the functions $\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \left( \frac{n \pi x}{L} \right) & 0 < x < L, \\ 0 & \text{otherwise}. \end{cases}$

$$\int_{-\infty}^{\infty} dx \, \psi_m^*(x) \psi_n(x) = \frac{2}{L} \int_{0}^{L} dx \, \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right).$$

To evaluate, observe that

$$\cos(x - \beta) = \cos x \cos \beta + \sin x \sin \beta$$
$$\cos(x + \beta) = \cos x \cos \beta - \sin x \sin \beta$$

$$\Rightarrow \cos(x - \beta) - \cos(x + \beta) = 2 \sin x \sin \beta.$$ 

Setting $\alpha = \frac{m \pi x}{L}$, $\beta = \frac{n \pi x}{L}$, the integral becomes

$$\int_{-\infty}^{\infty} dx \, \psi_m^*(x) \psi_n(x) = \frac{1}{L} \int_{0}^{L} dx \left( \cos \left( \frac{(m-n) \pi x}{L} \right) - \cos \left( \frac{(m+n) \pi x}{L} \right) \right)$$

$$= \left\{ \begin{array}{ll} \frac{1}{L} \left[ \frac{1}{(m-n)\pi} \sin \left( \frac{(m-n) \pi x}{L} \right) - \frac{1}{(m+n)\pi} \sin \left( \frac{(m+n) \pi x}{L} \right) \right]_{0}^{L} & m \neq n, \\ \frac{1}{L} \left[ \frac{1}{m\pi} \sin \left( \frac{2m \pi x}{L} \right) \right]_{0}^{L} & m = n, \end{array} \right.$$ 

$$= \left\{ \begin{array}{ll} \frac{1}{L} \left[ 0 - 0 \right] & m \neq n, \\ \frac{1}{L} \left[ 1 - 0 \right] & m = n, \end{array} \right.$$ 

$$= \left\{ \begin{array}{ll} 0 & m \neq n, \\ 1 & m = n. \end{array} \right.$$