\((\hat{A}, \hat{B})^+ = (\hat{A}\hat{B} - \hat{B}\hat{A})^+\)
\begin{align*}
= (\hat{A}\hat{B})^+ - (\hat{B}\hat{A})^+ \\
= \hat{B}^+\hat{A}^+ - \hat{A}^+\hat{B}^+ \\
= -(\hat{A}^+\hat{B}^+ - \hat{B}^+\hat{A}^+) \\
= -[\hat{A}^+\hat{B}^+] \\
\end{align*}

Can show:
\((i\hat{c})^+ = -i\hat{c}^+ \quad (i \rightarrow -i \text{ under Hermitian conjugation})\)

So,
\([\hat{A}, \hat{B}] = i\hat{c}\),
then taking the Hermitian conjugate of both sides, we obtain
\(-[\hat{A}^+\hat{B}^+] = -i\hat{c}^+.
\Rightarrow [\hat{A}, \hat{B}] = i\hat{c}^+.

If \(A, B\) are Hermitian, then
\(\hat{A}^+ = \hat{A}\)
\(\hat{B}^+ = \hat{B}\)
and then last result becomes,
\([\hat{A}, \hat{B}] = i\hat{c}\).

Compare to the definition \([\hat{A}, \hat{B}] = i\hat{c}\), then, we see that
\(\hat{c} = \hat{c}^+\)
\(\text{i.e. } \hat{c} \text{ is Hermitian.} \checkmark\)

Consistent with
\([\hat{x}, \hat{p}] = ik \Rightarrow \text{since } \hat{x}, \hat{p} \text{ are Hermitian,} \)
\(\text{so is } k. \ (k \text{ is a real number } \Rightarrow \text{trivially Hermitian})\)
\([\hat{L}_x, \hat{L}_y] = ik\hat{L}_z \Rightarrow \text{since } \hat{L}_x, \hat{L}_y \text{ are Hermitian,} \)
\(\text{so is } \hat{L}_z. \checkmark\)
Problem

Using the definition
\[ \int_{-\infty}^{\infty} dx \, \Psi^* \hat{A}^+ \Psi = \int_{-\infty}^{\infty} dx \, (\hat{A} \Psi)^* \Psi \]
determine the Hermitian conjugate \( \hat{P}^+ \) of the momentum operator \( \hat{P} = -ik \frac{\partial}{\partial x} \).
(Hint: you'll need to perform an integration by parts.)

Solution:
\[ \int_{-\infty}^{\infty} dx \, \Psi^* \hat{P}^+ \Psi = \int_{-\infty}^{\infty} dx \, (\hat{P} \Psi)^* \Psi \]
\[ = \int_{-\infty}^{\infty} dx \, (-ik \frac{\partial}{\partial x})^* \Psi \]
\[ = \int_{-\infty}^{\infty} dx \, ik \frac{\partial \Psi}{\partial x} \Psi \]
\[ = iK \Psi \Psi^* \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \, \Psi^* ik \frac{\partial \Psi}{\partial x} \]

Compare LHS and RHS, we see that
\[ \hat{P}^+ = -ik \frac{\partial}{\partial x} \]
do \( \hat{P}^+ = \hat{P} \), and \( \hat{P} \) is Hermitian.
(\( \uparrow \) \( \Rightarrow \) real eigenvalues)

Problem (a) What is \((\hat{A} \hat{B})^+\)?

(b) Deduce that if \( \hat{A}, \hat{B} \) are Hermitian, and \([\hat{A}, \hat{B}] = i\xi \), then \( \xi \) is also Hermitian.

a) \[ \int_{-\infty}^{\infty} dx \, \Psi^* \hat{A}^+ \Psi = \int_{-\infty}^{\infty} dx \, (\hat{A} \Psi)^* \Psi \]

Convenient Notation:
\[ \langle \Psi_1, \Psi_2 \rangle = \int_{-\infty}^{\infty} dx \, \Psi_1^* \Psi_2 \]

Then, definition becomes,
\[ \langle \Psi, \hat{A}^+ \Psi \rangle = \langle \hat{A} \Psi, \Psi \rangle \] \hspace{1cm} (1)

So,
\[ \langle \Psi, (\hat{A} \hat{B})^+ \Psi \rangle = \langle (\hat{A} \hat{B}) \Psi, \Psi \rangle\]
\[ = \langle \hat{A} (\hat{B} \Psi), \Psi \rangle \]
\[ = \langle (\hat{B} \Psi), \hat{A}^+ \Psi \rangle \] \hspace{1cm} (Rule (1) applied to \( \hat{B} \Psi, \Psi \), instead of \( \Psi, \Psi \))
\[ = \Psi, \hat{B}^+ (\hat{A}^+ \Psi) \rangle \]
\[ <\Psi, (\hat{A} \hat{B})^+ \Psi > = <\Psi, (\hat{B}^+ \hat{A}^+) \Psi > \]
and we can conclude
\[ (\hat{A} \hat{B})^+ = \hat{B}^+ \hat{A}^+ . \]
Solution

The eigenfunctions of \( \hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \) are \( \Phi_m(\phi) = N e^{i m \phi} \), satisfying

\[ \hat{L}_z \Phi_m = m \hbar \Phi_m. \]

For normalization on the unit circle, we require

\[ 1 = \int_0^{2\pi} d\phi \left| \Phi_m(\phi) \right|^2 = \int_0^{2\pi} d\phi N^2 = 2\pi N^2. \]

Therefore, \( N = \sqrt{\frac{1}{2\pi}} \) up to a choice of phase, and

\[ \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{i m \phi}. \]

(a) Method 1. The given wavefunction

\[ \psi(\phi) = \frac{1}{\sqrt{13\pi}} + \frac{1}{\sqrt{6\pi}} e^{i \phi} \]

can be written as

\[ \psi(\phi) = \sqrt{\frac{2}{3}} \Phi_0(\phi) + \sqrt{\frac{1}{3}} \Phi_1(\phi). \]

In general, if \( \psi(\phi) \) has an expansion \( \psi(\phi) = \sum \psi_m \Phi_m(\phi) \), then the probability of measuring \( L_z = m \hbar \) is

\[ P_m = |C_m|^2. \]

For the given wavefunction, we can read off

\[ C_0 = \frac{1}{\sqrt{3}}, \quad C_1 = \frac{1}{\sqrt{3}}, \quad C_m = 0 \quad \text{otherwise}. \]

Therefore,

\[ P_0 = |C_0|^2 = \frac{1}{3} \] (probability of measuring \( L_z = 0 \))
\[ P_1 = |C_1|^2 = \frac{1}{3} \] (probability of measuring \( L_z = \hbar \))
\[ P_m = |C_m|^2 = 0 \quad \text{otherwise}. \]

The total probability \( \sum_m P_m = \frac{2}{3} + \frac{1}{3} + 0 = 1 \) is indeed 1, so \( \psi(\phi) \) is properly normalized.
Method 2. By direct computation,

\[ \int d\phi \, \psi^* \psi = \int_{0}^{2\pi} d\phi \left( \frac{1}{13\pi} + \frac{1}{16\pi} e^{-i\phi} \right) \left( \frac{1}{13\pi} + \frac{1}{16\pi} e^{i\phi} \right) \]
\[ = \int_{0}^{2\pi} d\phi \left( \frac{1}{3\pi} + \frac{1}{6\pi} + \frac{1}{3\pi} \left( e^{i\phi} + e^{-i\phi} \right) \right) \]
\[ = \int_{0}^{2\pi} d\phi \left( \frac{1}{2\pi} + \frac{\sqrt{2}}{3\pi} \cos \phi \right) \]
\[ = \frac{1}{2\pi} (2\pi) + \frac{\sqrt{2}}{3\pi} (0) = 1, \]

as desired.

(b,c) From Part (a), Method 1, the possible results of measurement of \( L_z \) are \( L_z = 0 \) or \( L_z = \frac{\hbar}{\sqrt{2}} \), with probabilities \( \frac{2}{3} \) and \( \frac{1}{3} \), respectively.
Townsend 6.11. Eigenfunction of $\hat{L}_y$.

We are given the normalized angular wavefunction

$$\psi(\theta, \phi) = \sqrt{\frac{3}{8\pi}} (-\sin \theta \cos \phi + i \cos \theta)$$

and are asked to show that it is an eigenfunction of

$$\hat{L}_y = -i \hbar \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

and to compute the eigenvalue.

Solution

$$-i \cos \phi \frac{\partial}{\partial \theta} (-\sin \theta \cos \phi + i \cos \theta) = -i \cos \phi (-\cos \theta \cos \phi - i \sin \theta)$$
$$= -i \cos \theta \sin \phi - i \cos \theta \cos \phi,$$

$$i \cot \theta \sin \phi \frac{\partial}{\partial \phi} (-\sin \theta \cos \phi + i \cos \theta) = i \cot \theta \sin \phi \left( \sin \theta \sin \phi \right)$$
$$= i \cos \theta \sin^2 \phi,$$

so, adding and multiplying by $\hbar$, we have

$$\hat{L}_y (-\sin \theta \cos \phi + i \cos \theta) = \hbar \left( -\sin \theta \cos \phi + i \cos \theta \right),$$

and

$$\hat{L}_y \psi = \hbar \psi.$$

The given wavefunction $\phi$ is an eigenfunction of $\hat{L}_y$ with eigenvalue $\hbar$. This can also be seen as follows:

$$\psi(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \left( \frac{-x + i z}{r} \right)$$
$$= i \sqrt{\frac{3}{8\pi}} \left( \frac{z + i x}{r} \right),$$

using $z = r \cos \theta$, $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$.

This is a cyclic permutation $(x \rightarrow z, y \rightarrow x, z \rightarrow y)$ of

$$-i Y_1^l (\theta, \phi) = i \sqrt{\frac{3}{8\pi}} \left( \frac{x + iy}{r} \right).$$
Since $Y_l^m(\theta, \phi)$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $m\pi = l \pi$, it follows from the cyclic permutation that $\psi(\theta, \phi)$ is an eigenfunction of $\hat{L}_y$ with eigenvalue $\frac{\pi}{\hbar}$. 