Problem 2.20

When a pebble is tossed in a pond, a circular wave pulse propagates outward from the disturbance. In addition, surface ripples move inward relative to the outgoing pulse (but still outward in an absolute sense). Explain this effect in terms of the group and phase velocity, given that the phase velocity of the ripples is

\[
U_p = \frac{2\pi T}{\sqrt{\lambda \rho}}
\]

where \(T\) is the surface tension, \(\rho\) the density of the liquid, and \(\lambda\) is the wavelength.

Solution

\[
U_p = \frac{TK}{\sqrt{\lambda \rho}} \quad \text{since} \quad k = \frac{2\pi}{\lambda}.
\]

Also, in terms of \(\omega(k)\) and \(k\), \(U_p = \frac{\omega}{k}\). Therefore,

\[
\omega = kU_p
\]

\[
\omega = \frac{TK^3}{\sqrt{\lambda \rho}} \quad \text{(dispersion relation \(\omega = \omega(k)\))}
\]

The group velocity is

\[
U_g = \frac{d\omega}{dk} = \frac{d}{dk} \left( \sqrt{\frac{T}{\rho}} k^{3/2} \right) = \frac{3}{2} \sqrt{\frac{T}{\rho}} k^{1/2}
\]

\[
U_g = \frac{3}{2} \sqrt{\frac{TK}{\rho}}
\]

So, \(U_g = \frac{3}{2} U_p > U_p\).

As the envelope of the wave expands outward at speed \(U_g\), the smaller ripples fall behind. (The ripples move with speed \(U_p < U_g\).)
Problem 2.21.
Prove that the group velocity of a wavepacket is equal to the particle's velocity for a relativistic free particle.
(Recall: \( E = \hbar \omega = \sqrt{p^2 c^2 + m^2 c^4} \).)

Solution.
Dispersion relation \( \omega = \omega(k) \). Using \( p = \hbar k \), we have

\[
\omega(k) = \frac{1}{\hbar} \sqrt{p^2 c^2 + m^2 c^4} = \frac{1}{\hbar} \sqrt{\hbar^2 k^2 c^2 + m^2 c^4}.
\]

The group velocity is

\[
U_g = \frac{d\omega}{dk} = \frac{1}{\hbar} \frac{\frac{2\hbar k c^2}{\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}}}{\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}} = \frac{\hbar k c^2}{\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}}.
\]

To make this look nicer, write in terms of \( p = \hbar k \) and \( E = \hbar \omega = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4} \):

\[
U_g = \frac{pc^2}{E}.
\]

Does the right hand side agree with the particle's velocity \( U \) in special relativity? Recall, in terms of \( U \),

\[
E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - U^2/c^2}} \quad \text{and} \quad \vec{P} = \gamma m\vec{U} = \frac{m\vec{U}}{\sqrt{1 - U^2/c^2}}.
\]

Therefore, \( \frac{pc^2}{E} = \frac{\gamma m\vec{U} c^2}{\gamma mc^2} = U \), so indeed,

\[
U_g = U, \text{ the particle's velocity.}
\]

Comment: While we're on the subject, note that

\[
E = mc^2 \left(1 - \frac{U^2}{c^2}\right)^{-1/2}.
\]
Problem 2.23

Lasers can now be designed to emit pulses of light shorter than 30 microns long in the direction of their motion.

(a) Estimate the uncertainty in the momentum of the photon (in the direction of motion) in such a pulse.

(b) The momentum of a photon is \( p = \frac{h}{\lambda} \). Note typo in Townsend. Estimate the uncertainty in the wavelength of the pulse, assuming a nominal wavelength of 800 nm.

Solution

Heisenberg's uncertainty principle: \( \Delta x \Delta p_x = \frac{h}{2} \) \( (\Delta x \Delta k = \frac{1}{2}) \).

(a) \( \Delta x = 30 \times 10^{-6} \text{ m} \), \( \Delta p_x \approx \frac{h}{2 \Delta x} \).

We can estimate that the momentum uncertainty will be roughly of the order of the lower bound of the inequality:
\[ \Delta p_x \sim \frac{\hbar}{2\Delta x} = \frac{1.0 \times 10^{-34} \text{ J} \cdot \text{s}}{2(30 \times 10^{-6} \text{ m})} = 1.5 \times 10^{-30} \text{ kg m/s}. \]

(b) \[ p = \frac{\hbar}{\lambda} \implies \lambda = \frac{\hbar}{p}. \]

For small \( dp \), differentiation gives

\[ d\lambda = -\frac{\hbar dp}{p^2} = -\frac{\lambda^2}{\hbar} dp. \]

For small \( \Delta p \), we expect that the uncertainties in momentum \( \Delta p \) and wavelength \( \Delta \lambda \) will be related in the same way,

\[ \Delta \lambda = \frac{\lambda^2}{\hbar} \Delta p, \]

where we have omitted the minus sign, since the uncertainties \( \Delta p \) and \( \Delta \lambda \) are defined to be positive quantities.

Therefore, from the result of part (a),

\[ \Delta \lambda \sim \frac{(8.0 \times 10^{-7} \text{ m})^2}{1.0 \times 10^{-34} \text{ J} \cdot \text{s}} \times \frac{1.5 \times 10^{-30} \text{ kg m/s}}{1.5 \times 10^{-30} \text{ kg m/s}} = 9.6 \times 10^{-9} \text{ m} \sim 10 \text{ nm}. \]