Problem 3.2.

Show that there are no $E<0$ solutions to the time independent Schrödinger equation for a particle in a box.

Solution

The time-independent Schrödinger equation is

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x).$$

For a particle in a box, the potential is an infinite square well,

$$V(x) = \begin{cases} 0 & 0 < x < L, \\ \infty & \text{otherwise}. \end{cases}$$

Outside of $0 < x < L$, we have $\psi(x) = 0$. Inside, $V(x) = 0$, so the time-independent Schrödinger equation takes the form

$$\frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi \Rightarrow \frac{d^2 \psi}{dx^2} + \frac{2mE}{\hbar^2} \psi = 0.$$

We wish to explore the existence of $E<0$ solutions. For $E<0$, it is convenient to define

$$K = \sqrt{-\frac{2mE}{\hbar^2}}.$$

(This is a greek letter "kappa", not a latin "k").

Then,

$$\frac{d^2 \psi}{dx^2} - K^2 \psi = 0.$$

General solution: $\psi(x) = Ce^{Kx} + De^{-Kx}$,

or equivalently, $\psi(x) = A \sinh(Kx) + B \cosh(Kx)$,
Back to problem:

General $E<0$ solution to the $V(x)=0$ t-indep Schrödinger eq.:

$$\psi(x) = A\sinh(kx) + B\cosh(kx), \quad k = \sqrt{-\frac{2mE}{\hbar^2}} > 0.$$  

Impose boundary conditions (B.C.s) $\psi(0) = 0$ and $\psi(L) = 0$.

$$\psi(0) = 0 \implies A(0) + B(0) = 0 \quad \left( \text{since } \sinh(0) = 0 \text{ and } \cosh(0) = 1, \text{ i.e. sin, cos} \right)$$
$$\quad \text{so } B = 0 \text{ and } \psi(x) = A\sinh(kx).$$

Then, $\psi(L) = 0 \implies A\sinh(kL) = 0$.

No solution for $k>0$, $A \neq 0$.

Since $\sinh u$ (unlike $\sin \theta$) only vanishes at $u=0$.

So $A = 0$, and $\psi(x) = 0$.

We obtain only the trivial solution $\psi(x) = 0$. Therefore, the particle in a box has no $E<0$ stationary states.
Problem 3.4

The wavefunction for a particle in a box is
\[ \Psi(x, 0) = \sqrt{\frac{2}{a}} \psi_1(x) + \sqrt{\frac{1}{3}} \psi_2(x) \text{ at } t=0, \]

where \( \psi_1(x) \) and \( \psi_2(x) \) are the ground state and 1st excited state wavefunctions with energies \( E_1 \) and \( E_2 \).

What is \( \Psi(x, t) \)? What is the probability that a measurement of the energy yields the value \( E = E_n \)?

What is \( \langle E \rangle \)? How would you go about testing these predictions?

Solution

At time \( t \),
\[ \Psi(x, t) = \sqrt{\frac{2}{a}} \psi_1(x) e^{-i E_1 t/\hbar} + \sqrt{\frac{1}{3}} \psi_2(x) e^{-i E_2 t/\hbar}. \]

(We know that \( \psi_1(x, t) = \psi_1(x) e^{-i E_1 t/\hbar} \) and \( \psi_2(x, t) = \psi_2(x) e^{-i E_2 t/\hbar} \) are stationary states that solve the Schrödinger equation. By the principle of superposition, \( \sqrt{\frac{2}{a}} \psi_1(x, t) + \sqrt{\frac{1}{3}} \psi_2(x, t) \) is also a solution.)

\[
\text{Prob}(E_1) = |c_1|^2 = \frac{2}{3}, \quad \text{Prob}(E_2) = |c_2|^2 = \frac{1}{3}, \quad \text{Prob}(E > 2) = |c_0|^2 = 0.
\]

Note that total probability is \( \frac{2}{3} + \frac{1}{3} + 0 = 1 \).

\[ \langle E \rangle = \sum_{n=1}^{\infty} \text{Prob}(E_n) E_n = \sum_{n=1}^{\infty} |c_n|^2 E_n = \frac{2}{3} E_1 + \frac{1}{3} E_2. \]

To test these predictions, need to perform many \( E \) measurements on many systems, each prepared so that the wavefunction at \( t=0 \) is \( \Psi(x, 0) = \sqrt{\frac{2}{a}} \psi_1(x) + \sqrt{\frac{1}{3}} \psi_2(x) \). Then, check that \( E = E_1 \) \( \frac{1}{3} \) of the time, \( E = E_2 \) \( \frac{2}{3} \) of the time, and that the average value measured is \( \frac{2}{3} E_1 + \frac{1}{3} E_2 \).