Electromagnetism, Relativity & Particles — Homework Problems

1. **Reading Assignment**
   Read Good, Chapter 5.

2. **Ampère’s Law**
   (Good 3.13) Find the magnetic field \( B \) at a distance 3 cm from an extended plane carrying a uniform surface current with density \( j = 4 \text{ A/m} \). Also comment on the direction of \( B \).

3. **Ampère’s and Stokes’ Laws**
   (a) (Good 3.16) Find the magnetic field \( B(r) \) inside and outside of an infinite cylinder of radius \( R \) that is aligned along the \( z \) axis and carries a constant current density \( j = j \hat{e}_z \) in \( z \) direction (see Figure 3.32).
   
   (b) Find the corresponding vector potential \( A(r) \), using Stokes’ theorem:
   \[
   \Phi = \oint_{\text{loop}} A(r) \cdot \, \mathrm{d}r = \iint_{\text{enclosed area}} (\nabla \times A(r)) \cdot \hat{n}(r) \, \mathrm{d}a.
   \]
   (Note: \( \Phi \) is called the magnetic flux through the closed loop.)

4. **The Magnetic Monopole Field**
   Everybody knows that electric charges exist, but apparently there are no magnetic charges. People are still searching for them, even though nobody ever found one. In the following, you’re invited to show that if they exist, they have a very weird property: They come with strings attached!

   (a) The field \( B(r) \) of a static magnetic monopole with magnetic charge \( M \), placed at the origin, is given (in analogy to electric point charges \( Q \)) by:
   \[
   B(r) = \frac{\mu_0 M}{4\pi r^3}, \quad (r \neq 0).
   \]
   Show that it fulfills Maxwell’s equations \( \text{div} \, B(r) = 0 \) and \( \text{rot} \, B(r) = \mu_0 j(r) = 0 \).

   (b) Now determine a vector potential \( A(r) \) for the monopole field. Among the many choices for \( A(r) \), we pick one where \( A(r) = 0 \) on the positive \( z \) axis, and \( A(r) \) possesses rotational symmetry around this axis. Use Stokes’ theorem for a circular loop around the \( z \) axis, and pick a spherical segment (centered around the origin) as the enclosed surface (as shown in the figure on the back of the page). Verify that:
   \[
   A(r) = \frac{\mu_0 M}{8\pi r^2} \frac{1}{\cos^2(\frac{\theta}{2})} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.
   \]
   (Hints: To evaluate the area of the spherical “cap,” use spherical coordinates. You may need to use the trigonometric identities \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \), and \( 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \).)

   (c) Show that \( A(r) \) diverges near the negative \( z \) axis: A magnetic monopole is always associated with a line-like singularity in space — the simplest manifestation of a string. (A point charge like an electron in this sense is a point-like singularity in space.) — The magnetic field at the end of a very long, thin solenoidal coil is a good approximation to the magnetic monopole field (the longer and thinner, the better). Use the divergence theorem to show that the magnetic flux \( \Phi = \iint B \cdot \, \mathrm{d}a \) inside the solenoid must be \( \Phi = -\mu_0 M \).
loop

\( \vec{A} \)

spherical "cap" segment

\( r \)

\( \theta \)

\( \rho \)