Electromagnetism, Relativity & Particles — Homework Problems

1. Electromagnetic Field Tensor
The equations governing the electromagnetic potentials $A^\mu = (\Phi/c, A)$ read in 4–vector notation:

$$\partial_\mu \partial^\mu A_\nu - \partial_\nu \partial_\mu A^\mu = \partial_\mu F^{\mu\nu} = \mu_0 j_\nu,$$

where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor ($\mu, \nu = 0, 1, 2, 3$). Calculate the individual components of the field tensor and show that they indeed represent the electric and magnetic fields $E, B$.

2. Electromagnetic Waves
Consider an electromagnetic field that is described by the potentials:

$$\Phi(r, t) = 0, \quad A(r, t) = A_0 \cos(kz - \omega t),$$

where $A_0 = (A_{0x}, A_{0y}, A_{0z})$ is a constant amplitude vector, and $\omega$ and $k$ are constants.

(a) Determine the corresponding electric and magnetic fields $E(r, t), B(r, t)$.

(b) Determine the charge density and current density distributions $\rho(r, t), j(r, t)$ required to sustain this potential.

(c) Show: For a source-free solution with $\rho(r, t) = 0$ and $j(r, t) = 0$, the $z$–component $A_{0z}$ of the amplitude $A_0$ must vanish, and simultaneously $\omega = ck$ must hold. Describe this solution in words.

3. Plasma Waves
A neutral plasma is a gas that contains equal densities $n$ of positively charged and negatively charged carriers (with charges $+q$ and $-q$, and masses $m_+$ and $m_-$) that are allowed to move freely under the influence of electromagnetic fields. Good examples are ionized gases (like in the Earth ionosphere, a thin upper layer in the atmosphere, or solar matter), but also metals (in which typically one electron per atom moves around freely—this is why metals are good conductors of electricity). Plasmas support different kinds of waves, and here we examine two simple kinds of wavelike motion.

(a) An electric field $E$ accelerates the charges according to $F = \pm qE$. Show that the current density in the plasma is given by $j = nq(v_+ - v_-)$ (where $v_\pm$ are the velocities of both types of charges), and that the current density and the electric field are interrelated by:

$$\frac{\partial j}{\partial t} = nq^2 \left( \frac{1}{m_+} + \frac{1}{m_-} \right) E.$$

(b) Now take another look at the electromagnetic wave in Problem 2. Verify that there exist longitudinal solutions (called plasmons) for which the components $A_{0x}$ and $A_{0y}$ vanish, but not $A_{0z}$. Show that all these solutions share the same frequency $\omega = \omega_P$, independent of $k$, where $\omega_P$ denotes the plasma frequency:

$$\omega_P^2 = \frac{nq^2}{\epsilon_0} \left( \frac{1}{m_+} + \frac{1}{m_-} \right).$$

What is the magnetic field $B$ in this case?
(c) Besides the plasmons, there are also transversal wavelike solutions with $A_{0z} = 0$. Show that for these solutions, the frequency $\omega$ and the wave number $k$ are linked by the dispersion relation:

$$\omega(k)^2 = \omega_p^2 + \epsilon_0 k^2,$$

and demonstrate that the (phase) velocity $c(\omega)$ of the wave in the plasma is given by:

$$c(\omega) = \frac{\omega}{k} = \frac{\epsilon_0 \omega}{\sqrt{\omega^2 - \omega_p^2}}.$$

These waves can propagate in the plasma only for frequencies $\omega > \omega_p$. What is the index of refraction $n(\omega)$ of the plasma for these waves?

(d) Determine the plasma frequency $\omega_p$ in sodium metal (you will have to look up some physical properties of sodium). What would be the corresponding wavelength $\lambda$ of a wave traveling in vacuum with the same frequency? — The ionosphere becomes transparent for electromagnetic radiation at frequencies $\omega > \omega_p$, where the plasma frequency $\omega_p$ is around 30 MHz. What is the maximum density $n$ of free electrons in the ionosphere? (Note: It is safe to ignore the contribution of the positive ions. Why?)