Phys 306 - Lecture 1

Plan: Review Ch. 1, 2, 7, 8, 11, 12, 13
Roughly 2 problem sets per week (35%)
2 midterms (20% + 20%) + final (25%), take-home

Ch. 1: Infinite series, power series

Arithmetic sequence - difference $d$ is constant

\[
1, 3, 5, 7, 9, \ldots
d = 2
\]
\[
2, 10, 18, 26, \ldots
d = 8
\]
\[
a, a+d, a+2d, a+3d, \ldots
\]
\[
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow
\]
\[
a_1, a_2, a_3, a_4, \ldots
\]
\[
a_n = a + (n-1)d
\]

Geometric sequence - ratio $r$ is constant

\[
2, 4, 8, 16, 32, \ldots
r = 2
\]
\[
2, 6, 18, 54, 162, \ldots
r = 3
\]
\[
a, ar, ar^2, ar^3, \ldots
\]
\[
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow
\]
\[
a_1, a_2, a_3, a_4, \ldots
\]
\[
a_n = ar^{n-1}
\]

Series = Sum of terms in sequence

\[
S_N = a_1 + a_2 + a_3 + \ldots + a_N = \sum_{n=1}^{N} a_n
\]

Arithmetic series

\[
S_N = \frac{N}{2} (a_1 + a_N)
\]

(sum = \#terms \cdot average term)

Example (Gauss):

\[
1 + 2 + 3 + \ldots + 50 + 100 + 99 + 98 + \ldots + 51
\]
\[
= \frac{50}{2} \cdot (1 + 51)
\]
\[
= 101 \cdot \frac{101 - 1 + 101 + \ldots + 51}{50} = 50 \times 101.
\]

\text{fifty times}
Geometric series:

\[ S_N = a + ar + \cdots + ar^{N-1} = \frac{a(1-r^N)}{1-r} . \]

**Proof:** Problem 1.1.2 (p.3)

Infinite geometric series \( N \to \infty \)

- For \( |r| > 1 \), each term is bigger than previous one, sum diverges.
- For \( |r| = 1 \), sum either oscillates or diverges, doesn't converge.
- For \( |r| < 1 \), each term is smaller than previous one, sum has a chance of diverging.

Indeed, \( \lim_{N \to \infty} r^N = 0 \) for \( |r| < 1 \), so from formula above

\[ S = a + ar + ar^2 + \cdots = \lim_{N \to \infty} S_N = \frac{a}{1-r} . \]

More general series - specify a formula for an

**Example:** Boas 1.4.6

\[ S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \]

This sum "telescopes":

\[ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \]

So

\[ S = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = 1 \]
Convergence - many tests, see Boas.

For example, 1.6.1 The Ratio Test:

Define \( p_n = \frac{|a_{n+1}|}{a_n} \) and \( p = \lim_{n \to \infty} p_n \).

- If \( p > 1 \), the sequence approaches a "growing" geometric sequence, whose series diverges.
- If \( p = 1 \), use a different test.
- If \( p < 1 \), the sequence approaches a "shrinking" geometric sequence, whose series converges.

\[ \Rightarrow \text{If} \begin{cases} 
  p < 1, & \text{the series converges}, \\
  p = 1, & \text{use a different test}, \\
 p > 1, & \text{the series diverges}. 
\end{cases} \]

Example: Boas 1.6.18 \( \sum_{n=1}^{\infty} \frac{2^n}{n^2} \).

\[ p_n = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+1}/(n+1)^2}{2^n/n^2} \right| = \frac{2n^2}{(n+1)^2}, \]

\[ p = \lim_{n \to \infty} p_n = 2, \]

\( p > 1 \Rightarrow \text{series diverges} \).

1.10 Power series: infinite polynomials

\( \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \), or

\( \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + a_3 (x-a)^3 + \cdots \).
Interval of convergence — use ratio test to determine a range of x for which the power series converges.

Example: Boas 1.10.1 \( \sum_{n=0}^{\infty} (-1)^n x^n \).

\[ r_n = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \right| = |x|, \quad p = \lim_{n \to \infty} r_n = |x|. \]

Converges for \( p < 1 \implies |x| < 1. \)
Diverges for \( p > 1 \implies |x| > 1. \)

Check endpoints explicitly:

\( x = +1: \quad S = 1 - 1 + 1 - 1 + \cdots \) oscillatory (does not converge).
\( x = -1: \quad S = 1 + 1 + 1 + 1 + \cdots \) divergent.

Series does not converge at either endpoint

\( \implies \) interval of convergence \(-1 < x < 1.\)