Boas 8.2.10

$y' - xy = x, \quad y = 1 \text{ when } x = 0.$

$$\frac{dy}{dx} = (y+1)x$$

$$\int \frac{dy}{y+1} = \int x \, dx$$

$$\ln(y+1) = \frac{x^2}{2} + c$$

$$y+1 = Ae^{\frac{x^2}{2}}, \quad A = e^c$$

$$y = Ae^{\frac{x^2}{2}} - 1.$$  

Then, imposing the initial conditions $y = 1$ at $x = 0$, we have

$$1 = A - 1 \implies A = 2$$

$$y = 2e^{\frac{x^2}{2}} - 1$$

(See Mathematica printout for slope field and solution curves.)
Boas 3.2.19

\[
\frac{dI}{ds} = -\mu I, \\
\]

where

\[
\mu = \text{linear absorption coefficient} \\
\approx 10^{-2} \text{ ft}^{-1} \text{ for water.}
\]

Eq. (1) is separable:

\[
\int_{I_0}^{I} \frac{dI}{I} = - \int_{0}^{s} \mu ds \\
\ln I - \ln I_0 = -\mu s - 0, \quad \text{where } I_0 = \text{intensity at surface}
\]

\[
\ln \left( \frac{I}{I_0} \right) = -\mu s \\
I/I_0 = e^{-\mu s} \quad \iff \quad I = I_0 e^{-\mu s}
\]

At \( s = 1 \text{ ft} \), \( I/I_0 = e^{-\left(10^{-2} \text{ ft}^{-1}\right)(1 \text{ ft})} = e^{-10^{-2}} \approx 1 - 10^{-2} = 0.98 \).

Using \( e^x \approx 1 + x \) for \( \lvert x \rvert \ll 1 \).

At \( s = 50 \text{ ft} \), \( I/I_0 = e^{-\left(10^{-2} \text{ ft}^{-1}\right)(50 \text{ ft})} = e^{-1/2} = \frac{1}{\sqrt{e}} = 0.61 \).

At \( s = 500 \text{ ft} \), \( I/I_0 = e^{-\left(10^{-2} \text{ ft}^{-1}\right)(500 \text{ ft})} = e^{-5} = 6.74 \times 10^{-3} \).

At \( s = 1 \text{ mi} = 5280 \text{ ft} \), \( I/I_0 = e^{-\left(10^{-2} \text{ ft}^{-1}\right)(5280 \text{ ft})} = e^{-52.8} = 1.67 \times 10^{-23} \).
(b) Radioactive decay:

$$\frac{dN}{dt} = -\lambda N \iff N = N_0 e^{-\lambda t}$$

where \( \lambda \) is the decay constant.

The half-life \( T \) is defined by \( N(T) = \frac{1}{2} N_0 \):

$$\frac{1}{2} N_0 = N_0 e^{-\lambda T}$$

$$\frac{1}{2} = e^{-\lambda T}$$

$$-\ln 2 = -\lambda T \implies T = \frac{\ln 2}{\lambda}.$$
Boas 8.3.1

\[ y' + y = e^x \]  \hspace{1cm} (1)

The general solution to a linear first-order ODE \( y' + Py = Q \) is

\[ y = ce^{-I} + e^{-I} \int dx \quad e^x = Q, \] where \( I = \int Pdx \). (Boas 3.9)

For the given ODE (1),

\[ P = 1 \quad \text{and} \quad Q = e^x. \]

Therefore,

\[ I = \int Pdx = x, \]

\[ \int dx e^{-x} = \int dx e^x e^{-x} = \int dx e^{x-x} = \frac{1}{2} e^{2x}, \]

and

\[ y = ce^{-x} + e^{-x} \left( \frac{1}{2} e^{2x} \right) \]

\[ = ce^{-x} + \frac{1}{2} e^x, \]

which agrees with the solution obtained using DSolve[] in Mathematica. (See Mathematica printout.)
Boas 8.3.2

\[ x^2 y' + 3xy = 1 \iff y' + \frac{3}{x} y = \frac{1}{x^2}. \quad (1) \]

The general solution to \( y' + p(x)y = q(x) \) is given by

\[ y = ce^{-\int p(x)dx} + e^{-\int p(x)dx} \int dx e^{\int p(x)dx} q(x), \] where \( I = \int dx P. \) (Boas 3.9)

For the given ODE (1),

\[ P = \frac{3}{x} \text{ and } Q = \frac{1}{x^2}. \]

Therefore,

\[ I = \int \frac{3}{x} \, dx = 3 \ln x = \ln(x^3), \]

\[ \int dx e^{\int p(x)dx} = \int dx (x^3)(\frac{1}{x^2}) = \int dx x = \frac{x^2}{2}, \]

and

\[ y = \frac{C}{x^3} + \frac{1}{x^3} \left( \frac{x^2}{2} \right) = \frac{C}{x^3} + \frac{1}{2x}, \]

in agreement with the solution obtained using \texttt{DSolve[]} in \texttt{Mathematica}. (See Mathematica printout.)
Boas 8.4.11

\[ y' = \cos(x+y). \]

We are given the hint: Let \( u = x+y \); then \( u' = 1+y' \).

Using the hint, we have

\[ u' - 1 = \cos u, \quad \text{where} \quad u' = \frac{du}{dx} \]

\[ \frac{du}{dx} = 1+\cos u = 2\cos^2 \frac{u}{2} \]

\[ \int \frac{du}{2\cos^2 \frac{u}{2}} = \int dx \]

\[ \int \frac{du}{2 \sec^2 \frac{u}{2}} = \int dx \]

\[ \tan \frac{u}{2} = x + c, \quad \text{using} \quad \frac{d}{d\theta} \tan \theta = \sec^2 \theta \]

\[ u = 2 \arctan(x+c). \]

Then, since \( u = x+y \),

\[ y = -x + 2 \arctan(x+c). \]

This agrees with the solution obtained using `DSolve` in Mathematica, provided we identify \( \frac{1}{2} C17 \) there with \( c \) here. (See Mathematica printout.)
(* Boas 8.2.10. First define a "slope field" \(v(x,y)\) to be a vector field of unit vectors proportional to \((dx,dy)\), i.e., proportional to \((1,dy/dx) = (1,xy+y)\). *)

\[v[x_, y_] := Normalize[{1, x y + x}]\]

(* Now, plot the field of tangent vectors using the function VectorPlot. *)

\[\text{VectorPlot}[v[x, y], (x, -2, 2), (y, -2, 2)]\]

(* Boas 8.3.1 *)

\[\text{DSolve}[y'[x] + y[x] == E^x, y[x], x]\]
\[\{[y[x] \to \frac{e^x}{2} + e^{-x} C[1]]\}\]

(* Boas 8.3.2 *)

\[\text{DSolve}[x^2 y'[x] + 3 x y[x] == 1, y[x], x]\]
\[\{[y[x] \to \frac{1}{2 x} + \frac{C[1]}{x^3}]\}\]

(* Boas 8.4.11 *)

\[\text{DSolve}[y'[x] == \cos(x + y[x]), y[x], x]\]

\[\text{Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.} \]
\[\{[y[x] \to -x - 2 \arctan\left(\frac{1}{2} (-2 x - C[1])\right)], [y[x] \to -x + 2 \arctan\left(\frac{1}{2} (2 x + C[1])\right)]\}\]