Advanced Classical Mechanics — Homework Problems

1. Rotation of a Plane Cartesian Coordinate System

This problem deals with cartesian coordinate systems and rotations in two dimensions. We’ll also casually introduce some of the concepts of orthogonal matrices and rotation groups.

(a) The figure below shows two cartesian coordinate systems that have the same origin, but are rotated with respect to each other by an angle \( \theta \):

![Cartesian Coordinate Systems](image)

Show that the coordinates of a point in the rotated system \((x', y')\) are given by the following linear function of the original coordinates \((x, y)\):

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \cos \theta + y \sin \theta \\ y \cos \theta - x \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = D(\theta) \begin{pmatrix} x \\ y \end{pmatrix} .
\]

\( D(\theta) \) is known as a rotation matrix in two dimensions. Check that the matrix is orthogonal, i.e., the column vectors of this matrix are of unit length and perpendicular to each other. What is the determinant of the matrix?

(b) Verify that the product of two rotation matrices \( D(\theta_1)D(\theta_2) \) is in itself a rotation matrix. Show that the resulting angle is simply given by \( \theta = \theta_1 + \theta_2 \). Give a geometrical interpretation.

(c) In group theory, a group \( \mathcal{G} \) is defined as a collection of objects \( \{g\} \) (“group elements”), together with a group operation \( \circ \) that defines a “product” within the group \( g_i \circ g_j \) that obeys the following four rules:

i. Completeness: For all pairs of objects \( g_i \) and \( g_j \) in \( \mathcal{G} \), their product \( g_i \circ g_j \) is itself a member of the group.

ii. Associativity: For any three group elements \( g_i, g_j, g_k \), the relation holds:

\[
(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)
\]

iii. Neutral Element: The group \( \mathcal{G} \) contains one special member \( e \), the neutral element, that leaves all elements invariant under the product:

\[
e \circ g = g \circ e = g \quad \text{for all } g \in \mathcal{G}
\]

iv. Inverse Element: For each group element \( g \) there exists one inverse element \( g^{-1} \) so that

\[
g^{-1} \circ g = g \circ g^{-1} = e
\]
Show that if the product $\odot$ denotes the product of two matrices, the set of all rotation matrices $D(\theta)$ forms a group, the special orthogonal group in two dimensions, also known as $SO(2)$. Is the group commutative, i.e., does $g_i \circ g_j = g_j \circ g_i$ hold for any two group elements? Can you think of a non-commutative group?

(e) The rotation group is two-dimensional (it involves $2 \times 2$-matrices). Find an isomorphic group (i.e., a group with a product that follows exactly the same rules) that is one-dimensional. (Hint: Think of complex numbers!)

(f) Show that the rotation matrices $D(\theta)$ can be written in exponential form: $D(\theta) = \exp(\theta \cdot T)$, where the matrix $T$ is called the generator of the group $SO(2)$:

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(Hint: For the proof, use either the Taylor expansion for the exponential, or find a simple differential equation for $D(\theta)$.) Is the matrix $T$ itself a member of the group?

2. Metrics on a Sphere

Coordinates $(\theta, \phi)$ on a sphere of radius $R$ are customarily introduced as the polar angle $\theta$ between the point and the pole, and the azimuthal angle $\phi$ that represents its relative angle around the “equator” of the sphere. (Think of geographical latitude and longitude — the angle $\theta$ is the latitude, except that it is measured from the pole, not the equator; $\phi$ represents the longitude.)

(a) Show (geometrically): We may align a three-dimensional cartesian coordinate system so that the positive $z$ axis points to the pole of the sphere, and that its “meridian” of zero longitude is located in the $x-z$ plane. The coordinates $x, y, z$ of a point on the sphere read as a function of $\theta$ and $\phi$:

$$x(\theta, \phi) = R \sin \theta \cos \phi, \quad y(\theta, \phi) = R \sin \theta \sin \phi, \quad z(\theta, \phi) = R \cos \theta.$$  

(b) Prove that the metric (i.e., the length of an infinitesimal curve) on the sphere is given by:

$$(ds)^2 = R^2 (d\theta)^2 + R^2 \sin^2 \theta (d\phi)^2.$$  

Argue geometrically, but also verify this result by calculation using Eq. (3).

(Note: There is no two-dimensional cartesian coordinate system on the sphere — the sphere is a surface with non-Euclidean geometry. [You can verify this statement by looking at the sum of the three angles in a triangle on the sphere.] What we did here was, in math-speak, to embed the non-Euclidean spherical surface into a three-dimensional Euclidean space.)

(c) Express the metric $(ds)^2$ in three-dimensional spherical coordinates $(r, \theta, \phi)$. (That means that we allow the radius $r$ to vary now.) Give only the result and a brief explanation.

3. The Millikan Experiment

A charged droplet (charge $q$, mass $m$) is resting between two capacitor plates when at time $t = 0$, suddenly an alternating electric field $E(t) = E_0 \sin(\omega t)$ is applied. Apart from the electric force, the droplet is also subject to a viscous drag force proportional to the particle velocity $v$ that acts opposite to it, $F_{\text{drag}} = -\gamma v$. (Ignore gravity.)

(a) Set up a differential equation for $v(t)$.

(b) What is the particle velocity $v(t)$ for $t \geq 0$? (Note: The differential equation for $v(t)$ is of the inhomogeneous linear type and can be solved in two steps: First, omit the electric force, and verify that friction then causes any initial velocity to drop exponentially, $v(t) = A \exp(-\gamma t/m)$. In the second step, replace the constant $A$ by a time-dependent function $A(t)$ [variation of constants], and put the expression back into the differential equation. You will end up with an expression for $dA(t)/dt$.)

(c) What is the displacement of the particle $r(t)$? Plot a graph of both $r(t)$ and the electric field $E(t)$.