Physics Department
Bryn Mawr College
Physics 308

Fall 2005
Assignment #7
November 3, 2005

Advanced Classical Mechanics — Homework Problems

1. *A Bifurcation Problem* (30 P.)

   In various problems, the potential energy $U(x;\lambda)$ for a particle with mass m depends on some external parameter λ. A particularly simple example with interesting properties is the model potential:

   $$U(x;\lambda) = \alpha x^4 + \lambda x^2 \ , \tag{1}$$

   where $\alpha > 0$ is a positive constant.

   (a) Plot schematically the energy landscape for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. (3 P.)

   (b) The particle is now moving in the potential $U(x;\lambda)$ with total (conserved) energy E. Identify the equilibrium points of the particle in all three cases. What is their corresponding energy? Which ones correspond to stable equilibrium? (5 P.)

   (c) Sketch a *bifurcation diagram* that plots the position of the equilibrium points in the (x,λ) plane. Indicate also their character. What happens at the origin $(x = \lambda = 0)$? (5 P.)

   (d) In the vicinity of the stable equilibrium points, the particle may undergo small oscillations. Calculate their frequency ω as a function of the parameter λ. Make a plot of $\omega(\lambda)$. (Result: (7 P.)

   $$\omega(\lambda) = \begin{cases} 2\sqrt{-\lambda/m} & (\lambda < 0)\ , \\ \sqrt{2\lambda/m} & (\lambda > 0)\ ,) \end{cases} \tag{2}$$

   (e) Show that the period T of the oscillation is inversely proportional to the fourth root of the energy, $T(E) \propto E^{-1/4}$, when $\lambda = 0$. (*Hint*: Use the results from Problem 3, Assignment #6.) (3 P.)

   (f) Make a qualitative sketch of the oscillation period $T(E)$ as a function of E for all three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Identify and discuss the distinguishing features of each of these three curves. (7 P.)

2. *Free Particle as Central Force Problem* (20 P.)

   If the potential energy $U(\mathbf{r})$ everywhere vanishes, a particle of mass m and velocity v_0 will move freely according to Newton's first law. Because a vanishing force field is a particular case of a central force field (for any choice of the "force center"), the corresponding formalism must yield uniform motion along a straight line. Let's verify that in the following.

   (a) Express the velocity v_0 and the impact parameter b (the closest distance of the particle to the force center) in terms of the energy E and the angular momentum L. (3 P.)

   (b) Set up the radial equation for $r(t)$ and solve it. If the closest approach takes place at time $t = 0$ and at polar angle ϕ_0, show that the result can be written in the form: (5 P.)

   $$r(t) = \sqrt{b^2 + v_0^2 t^2} \ , \tag{3}$$

   (c) Integrate to find the temporal variation of the polar angle $\phi(t)$, and verify that: (5 P.)

   $$\phi(t) = \phi_0 + \arctan\left(\frac{v_0 t}{b}\right) \ , \tag{4}$$

— Please turn —

(d) Finally, solve the differential equation for the orbit $r(\phi)$ and show that: (5 P.)

$$r(\phi) = \frac{b}{\cos(\phi - \phi_0)} \ , \tag{5}$$

(e) Verify geometrically that (3)–(5) indeed describe uniform motion along a straight line. (2 P.)