Physics Department
Bryn Mawr College
Physics 308

Fall 2005
Assignment #8
November 17, 2005

Advanced Classical Mechanics — Homework Problems

1. *The Lenz Vector* (20 P.)

Interestingly, the shape of the Kepler orbits can be determined without any recourse to differential equations. To this end, we exploit a peculiarity of the potential $U(\mathbf{r}) = \alpha/r$, and first show that besides the angular momentum $\mathbf{L}$, there exists another conserved vector, the *Lenz vector* $\mathbf{Z}$:

$$\mathbf{Z} = \frac{1}{\alpha}\big(\dot{\mathbf{r}} \times \mathbf{L}\big) + \frac{\mathbf{r}}{r} \ . \qquad (1)$$

(a) Verify that $\mathbf{Z}$ is a constant of the motion, i.e., $\dot{\mathbf{Z}} = \mathbf{o}$ holds. (*Hint*: $\mathbf{L} = m(\mathbf{r} \times \dot{\mathbf{r}})$ is conserved. Show that $\mathrm{d}r/\mathrm{d}t = (\mathbf{r} \cdot \dot{\mathbf{r}})/r$, and finally use the rule for triple products from Assignment #1, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.) (6 P.)

(b) Show that $\mathbf{Z}$ is contained in the plane of the orbit. (*Hint*: Show that $\mathbf{L} \perp \mathbf{Z}$.) (3 P.)

(c) Show that $\mathbf{Z}$ points along the line connecting origin and pericenter of the motion (i.e., for bound orbits, along the major axis of the ellipse). (*Hint*: Verify that $\mathbf{Z} \times \mathbf{r} = \frac{1}{\alpha} r\dot{r}\mathbf{L}$. Where does this expression become zero?) (4 P.)

(d) Now form the scalar product $\mathbf{r} \cdot \mathbf{Z}$. Show that: (4 P.)

$$\mathbf{r} \cdot \mathbf{Z} = \frac{L^2}{m\alpha} + r \ . \qquad (2)$$

(e) Derive the equation of the orbit $r(\phi)$ from Eq. (2), and show that the magnitude of the Lenz vector equals the eccentricity of the orbit, $\varepsilon = |\mathbf{Z}|$. (3 P.)

2. *Charge in Electric Field of a Wire* (35 P.)

Consider a thin, straight, uniformly charged long wire with charge density (charge per length) σ. Such a wire produces a static electric field with cylindrical symmetry. If the wire is placed along the z axis of the coordinate system, it will exert a force $\mathbf{F}(\mathbf{r})$ on a charged particle (mass m, charge q) moving outside the wire that is given by:

$$\mathbf{F}(\mathbf{r}) = \frac{\sigma q}{2\pi\epsilon_0}\,\frac{1}{\rho^2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \ , \qquad (3)$$

where $\rho = (x^2 + y^2)^{1/2}$ denotes the radial distance of the charge to the wire. In the following, we examine the motion of the charge.

(a) Verify that the force field $\mathbf{F}(\mathbf{r})$ is conservative, i.e., the work ΔW along any closed path vanishes. (Why is this even true for trajectories that loop around the z axis?) (5 P.)

(b) Any conservative force field can be cast as the gradient of a suitable potential function, $\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla} U(\mathbf{r})$. Using some arbitrarily chosen radial distance ρ_0 as a reference point, show that the potential energy in the field of the wire can be written as: (5 P.)

$$U(\rho) = -\frac{\sigma q}{2\pi\epsilon_0}\,\ln\left(\frac{\rho}{\rho_0}\right) \ . \qquad (4)$$

Sketch this potential both for attractive and repulsive interaction.

— Please turn —

(c) Show that the momentum component of the charge in z direction is conserved, $p_z = \text{const}$. Find $z(t)$. (2 P.)

(d) Show that the z component of the *angular momentum* is also conserved, $L_z = \text{const}$. (3 P.)

(e) Use conservation of energy to derive a *radial equation* for the distance $\rho(t)$ (akin to the radial equation in the central force problem): (6 P.)

$$\frac{m\dot{\rho}^2}{2} + U_{\text{eff}}(\rho) = \frac{m\dot{\rho}^2}{2} + \frac{L_z^2}{2m\rho^2} + U(\rho) = E_\rho \ . \tag{5}$$

Here, $E_\rho = E - p_z^2/(2m)$ is a conserved "radial" energy. Sketch the effective potential $U_{\text{eff}}(\rho)$ as a function of ρ both in the attractive and repulsive case. In which respect does the attractive case fundamentally differ from the Kepler problem?

(f) Now consider only attractive interaction. For given L_z, determine the equilibrium point ρ_{min} in the potential $U_{\text{eff}}(\rho)$. Why does it correspond to a circular orbit around the wire? What is the energy E_{min} of the charge for circular motion? Show that the period of revolution T_{rev} of the charge grows linearly with L_z: (5 P.)

$$T_{\text{rev}} = -\frac{4\pi^2\epsilon_0 L_z}{\sigma q} \ . \tag{6}$$

This implies that the velocity v_{circ} of a charge moving on a circular orbit does *not* depend on its radius ρ_{min}. What is v_{circ}?

(g) The radial equation (5) cannot be integrated in closed form. However, for energies E_ρ close to E_{min}, the variation in ρ is small, and the potential nearly parabolic. Use the theory of small oscillations to show that $\rho(t)$ varies approximately harmonically around the minimum ρ_{min}, with an angular frequency: (5 P.)

$$\omega_{\text{radial}} = \frac{\sigma q}{\sqrt{2}\,\pi\epsilon_0 L_z} \ . \tag{7}$$

(h) Now consider the shape of these "almost circular" orbits. Show that in the limit $E \to E_{\text{min}}$, the angle of the orbit $\phi(\rho)$ progresses during one complete radial oscillation by an angle $\Phi = \sqrt{2}\,\pi$. Are these orbits closed? Sketch an "almost circular" orbit in the field of the wire. (4 P.)