Advanced Classical Mechanics — Homework Problems

1. The Lenz Vector

Interestingly, the shape of the Kepler orbits can be determined without any recourse to differential equations. To this end, we exploit a peculiarity of the potential $U(r) = \alpha/r$, and first show that besides the angular momentum $L$, there exists another conserved vector, the Lenz vector $Z$:

$$Z = \frac{1}{\alpha} (\dot{r} \times L) + \frac{r}{r}.$$  \hspace{1cm} (1)

(a) Verify that $Z$ is a constant of the motion, i.e., $\dot{Z} = 0$ holds. (Hint: $L = m(r \times \dot{r})$ is conserved. (6 P.)

(b) Show that $Z$ is contained in the plane of the orbit. (Hint: Show that $L \perp Z$.)

(c) Show that $Z$ points along the line connecting origin and pericenter of the motion (i.e., for bound orbits, along the major axis of the ellipse). (Hint: Verify that $Z \times r = \frac{1}{\alpha} r/\dot{r} L$. Where does this expression become zero?)

(d) Now form the scalar product $r \cdot Z$. Show that:

$$r \cdot Z = \frac{L^2}{m\alpha} + r.$$ \hspace{1cm} (2)

(e) Derive the equation of the orbit $r(\phi)$ from Eq. (2), and show that the magnitude of the Lenz vector equals the eccentricity of the orbit, $\varepsilon = |Z|$.

2. Charge in Electric Field of a Wire

Consider a thin, straight, uniformly charged long wire with charge density (charge per length) $\sigma$. Such a wire produces a static electric field with cylindrical symmetry. If the wire is placed along the $z$ axis of the coordinate system, it will exert a force $F(r)$ on a charged particle (mass $m$, charge $q$) moving outside the wire that is given by:

$$F(r) = \frac{\sigma q}{2\pi\epsilon_0} \frac{1}{\rho^2} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$ \hspace{1cm} (3)

where $\rho = (x^2 + y^2)^{1/2}$ denotes the radial distance of the charge to the wire. In the following, we examine the motion of the charge.

(a) Verify that the force field $F(r)$ is conservative, i.e., the work $\Delta W$ along any closed path vanishes. (Why is this even true for trajectories that loop around the $z$ axis?)

(b) Any conservative force field can be cast as the gradient of a suitable potential function, $F(r) = -\nabla U(r)$. Using some arbitrarily chosen radial distance $\rho_0$ as a reference point, show that the potential energy in the field of the wire can be written as:

$$U(\rho) = -\frac{\sigma q}{2\pi\epsilon_0} \ln \left( \frac{\rho}{\rho_0} \right).$$ \hspace{1cm} (4)

Sketch this potential both for attractive and repulsive interaction.

— Please turn —
(c) Show that the momentum component of the charge in $z$ direction is conserved, $p_z = \text{const.}$ Find $z(t)$.  

(d) Show that the $z$ component of the angular momentum is also conserved, $L_z = \text{const.}$  

(e) Use conservation of energy to derive a radial equation for the distance $\rho(t)$ (akin to the radial equation in the central force problem):

$$\frac{m\rho^2}{2} + U_{\text{eff}}(\rho) = \frac{m\rho^2}{2} + \frac{L_z^2}{2m\rho^2} + U(\rho) = E_{\rho}.$$  

(5)

Here, $E_{\rho} = E - p_z^2/(2m)$ is a conserved “radial” energy. Sketch the effective potential $U_{\text{eff}}(\rho)$ as a function of $\rho$ both in the attractive and repulsive case. In which respect does the attractive case fundamentally differ from the Kepler problem?

(f) Now consider only attractive interaction. For given $L_z$, determine the equilibrium point $\rho_{\text{min}}$ in the potential $U_{\text{eff}}(\rho)$. Why does it correspond to a circular orbit around the wire? What is the energy $E_{\text{min}}$ of the charge for circular motion? Show that the period of revolution $T_{\text{rev}}$ of the charge grows linearly with $L_z$:

$$T_{\text{rev}} = -\frac{4\pi^2\epsilon_0 L_z}{\sigma q}.$$  

(6)

This implies that the velocity $v_{\text{circ}}$ of a charge moving on a circular orbit does not depend on its radius $\rho_{\text{min}}$. What is $v_{\text{circ}}$?

(g) The radial equation (5) cannot be integrated in closed form. However, for energies $E_{\rho}$ close to $E_{\text{min}}$, the variation in $\rho$ is small, and the potential nearly parabolic. Use the theory of small oscillations to show that $\rho(t)$ varies approximately harmonically around the minimum $\rho_{\text{min}}$ with an angular frequency:

$$\omega_{\text{radial}} = \frac{\sigma q}{\sqrt{2\pi}\epsilon_0 L_z}.$$  

(7)

(h) Now consider the shape of these “almost circular” orbits. Show that in the limit $E \rightarrow E_{\text{min}}$, the angle of the orbit $\phi(\rho)$ progresses during one complete radial oscillation by an angle $\Phi = \sqrt{2}\pi$. Are these orbits closed? Sketch an “almost circular” orbit in the field of the wire.