Physics 325: General Relativity
Spring 2012
Problem Set 2

Due: Thu 2 Feb 2012.

Reading: Please skim Chapter 3 in Hartle. Much of this should be review, but probably not all of it—be sure to read Box 3.2 on Mach’s principle. Then start on Chapter 6.

Problems:


3. Lorentz transformations and hyperbolic geometry. In class, we saw that a Lorentz transformation in 2D can be written as $a'^\alpha = L_{\alpha \beta}(\vartheta)a^\beta$, that is,

$$
\begin{pmatrix}
    a^0' \\
    a^1'
\end{pmatrix} =
\begin{pmatrix}
    \cosh \vartheta & -\sinh \vartheta \\
    -\sinh \vartheta & \cosh \vartheta
\end{pmatrix}
\begin{pmatrix}
    a^0 \\
    a^1
\end{pmatrix},
$$

where $a$ is spacetime vector. Here, the rapidity $\vartheta$ is given by

$$
\tanh \vartheta = \beta, \quad \cosh \vartheta = \gamma, \quad \sinh \vartheta = \gamma\beta,
$$

where $v = \beta c$ is the velocity of frame $S'$ relative to frame $S$.

(a) Show that two successive Lorentz boosts of rapidity $\vartheta_1$ and $\vartheta_2$ are equivalent to a single Lorentz boost of rapidity $\vartheta_1 + \vartheta_2$. In other words, check that $L^\alpha_{\gamma}(\vartheta_1)L(\vartheta_2)^\gamma_\beta = L^\alpha_{\beta}(\vartheta_1 + \vartheta_2)$, where $L^\alpha_\beta(\vartheta)$ is the matrix in Eq. (1). You will need the following hyperbolic trigonometry identities:

$$
\cosh(\vartheta_1 + \vartheta_2) = \cosh \vartheta_1 \cosh \vartheta_2 + \sinh \vartheta_1 \sinh \vartheta_2,
\sinh(\vartheta_1 + \vartheta_2) = \sinh \vartheta_1 \cosh \vartheta_2 + \cosh \vartheta_1 \sinh \vartheta_2.
$$

(b) From Eq. (3), deduce the formula for $\tanh(\vartheta_1 + \vartheta_2)$ in terms of $\tanh \vartheta_1$ and $\tanh \vartheta_2$. For the appropriate choice of $\vartheta_1$ and $\vartheta_2$, use this formula to derive the special relativistic velocity tranformation rule

$$
V' = \frac{V - v}{1 - vv/c^2}.
$$
You will need to use the relation (2) between rapidity and velocity. Here, \( V \) is the velocity of a particle in frame \( S \), \( v \) is velocity of frame \( S' \) relative to frame \( S \), and \( V' \) is the velocity of the particle in frame \( S' \). (See Hartle Eq. (4.28a).)

(c) Optional (extra credit). Hartle Problem 5.2. This problem shows that the spacetime analog of the Euclidean result \( \vec{a} \cdot \vec{b} = ab \cos \theta_{ab} \) is

\[
\vec{a} \cdot \vec{b} = -ab \cosh \vartheta_{ab}
\]  

provided \( a \) and \( b \) are timelike.

4. **Uniform proper acceleration.** Hartle Problem 5.6. In addition, please add the following:

(e) Compute the acceleration in a frame in which the particle is instantaneously at rest. You should find that the result of is independent of time. Hence, this motion is often referred to as uniform proper acceleration. It is the most natural definition of “uniform acceleration” in special relativity.

5. **Cosmic rays and the GZK cutoff.** Please answer the following questions based on Box 5.1 in Hartle and the Physics Today article by Bertram Schwarzschild handed out in class. (One sentence answers suffice.)

(a) What is the GZK cutoff? (b) What happens to a cosmic ray proton that has an energy above the GZK cutoff? (c) Have we observed cosmic rays with energies above the GZK cutoff? (d) Please describe what the Pierre Auger Observatory is, and (e) summarize the results reported in the Physics Today article. (f) Do the results agree with our expectations?

In case you are curious what active galactic nuclei (AGN) are, they’re described on p. 288 in Hartle.