Physics 325: General Relativity
Spring 2012

Problem Set 3

Due: Thu 9 Feb 2012.

Reading: Please finish reading Chapter 6 in Hartle. In addition, please read Chapter 2. Box 2.3 on Map Projections is more relevant to general relativity than you might think. To represent the causal structure of infinite spacetime on a finite sheet of paper, we will use *Penrose diagrams*, which are equiangular projections of spacetime.

Problems:

1. **Aberration.** Hartle Problem 5.16.

2. **Relativistic Beaming.** Hartle Problem 5.17. Hints:

   To set up part (a), choose axes so that in the source frame $S$ the wave 3-vector $\vec{k}$ is lies the $xy$ plane and makes an angle $\alpha$ with the observer velocity $\vec{V} = V \hat{x}$. To find the new angle $\alpha'$ in the observer frame $S'$, transform the components of the wave 4-vector.

   In part (b), the solid angle between two cones of angle $\alpha$ and $\alpha + d\alpha$ is $d\Omega = 2\pi \sin \alpha d\alpha$. Solid angle refers to the area on the unit sphere ($\int \sin \theta d\theta d\phi$ in spherical coordinates), just as ordinary angle refers to the length on the unit circle ($\int d\phi$ in polar coordinates.)

3. **Frequency as measured by an accelerated observer.**

   Hartle Problem 5.18. Examples 5.9 uses a “trick” to find the shifted frequency of a photon as measured by an accelerated observer. This problem asks you to check the trick by using Lorentz transformations, or equivalently, the Doppler shift formula, Eq. (5.73).

4. **Lagrangian and Hamiltonian in special relativity.**

   In recitation section, we argued that the action for a massive particle in special relativity is $S = -mc^2 \int_{\tau_i}^{\tau_f} \, d\tau$, where $\tau$ is the proper time. Different parameterizations of the worldline give different Lagrangians. Hartle uses an arbitrary parameter $\sigma$ in Sec. 5.4. In recitation
section we instead specialized to a particular Lorentz frame and used the time \( t \) in that frame. In this case, \( S = \int_{t_i}^{t_f} dt \mathcal{L}, \) where

\[
\mathcal{L}(\dot{x}) = -mc^2 \sqrt{1 - \dot{x}^2/c^2}.
\]  

(a) What is the generalized momentum \( p \) in the \( x \) direction?

(b) What is the Euler-Lagrange equation for \( x \)?

(c) The Hamiltonian \( H(x, p, t) = p\dot{x} - \mathcal{L} \) gives the conserved energy of the particle. Show that

\[
H = \frac{mc^2}{\sqrt{1 - v^2/c^2}}.
\]  

Here, \( v = v(p) \) is understood to be a function of the momentum. (By definition, the Lagrangian is a function of \( x, \dot{x}, t \) and \( H \) is a function of \( x, p, t \).)

(d) To find an explicit expression for \( H \) in terms of the momentum, first show that

\[
H^2 - p^2 c^2 = m^2 c^4,
\]  

from parts (a) and (c). Then,

\[
H(p) = \sqrt{p^2 c^2 + m^2 c^4}.
\]  

(e) If the motion is vertical near the earth’s surface (with \( x \) increasing downward), then we need to account for gravity. In the approximation that is \( g \) constant, the Hamiltonian becomes

\[
H(x, p) = \sqrt{p^2 c^2 + m^2 c^4} - mgx.
\]  

Hamilton’s equations of motion are \( \dot{x} = \partial H/\partial p \) and \( \dot{p} = -\partial H/\partial x \). What are the resulting equations in this case? What is the limiting value of \( \dot{x} \) as \( p \to \infty \)?

(f) Solve for \( p = p(t) \). In natural units \((c = 1)\), show that the equation for \( \dot{x} \) becomes

\[
\frac{dx}{dt} = \frac{gt}{\sqrt{1 + g^2 t^2}}.
\]  

Does this look familiar from last week’s problem set?