Physics 325: General Relativity
Spring 2012
Problem Set 8

Due: Thu 29 March 2012.

Reading: Please read Sec. 11.2 on accretion disks and the first half of Chapter 12. For the following week, please finish reading Chapter 12 and read Chapter 13. We will not discuss Chapter 13 in class, except for Sec. 13.3 on Hawking radiation.

Announcement: Review Problem Set 2 will be distributed on Thursday 29 March and due Thursday 6 April. It will cover the material in Chapters 7–9 and on Problem Sets 5–8.

Problems (continued on other side):

1. Orbits with $E$ near the maximum of the effective potential. Hartle Problem 9.5. For the case that $E$ is slightly below the maximum of $V_{\text{eff}}$, please also plot the orbit using the Mathematica program *Shape of Orbits in the Schwarzschild Geometry*, on the textbook website. A local copy can be found at:

   http://www.brynmawr.edu/physics/courses/phys325/spring12/Mathematica/schorbits.nb

   Use the values $\ell = 4.3$ and $E = 0.0401300$ (both positive). In more recent versions of Mathematica, you might need to change

   MaxBend → .1, PlotDivision → 50,

   to

   Method → {MaxBend → .1, PlotDivision → 50},

   in the last statement of the file, to avoid errors. Please briefly describe the resulting orbit.

2. Precession of the perihelion of a planet. Hartle Problem 9.15. In this problem, we will derive Eq. (9.55) in Hartle,

   $\delta \phi_{\text{prec}} = 6\pi \left(\frac{GM}{c^2\ell}\right)^2$ (to first order in $1/c^2$). (1)
As indicated in the textbook, the expansion is a bit tricky. The naive expansion of the square root in the denominator of Eq. (9.52) in the small quantity \((2GM\ell^2)/(c^2r^3)\) gives an infinite integral.

**Hints:**

(a) Start with

\[
\Delta \phi = 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2} \left(1 - \frac{2GM}{c^2r}\right)^{1/2} \left[\frac{c^2}{2} \left(1 - \frac{2GM}{c^2r}\right) - \left(c^2 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}.
\]  

Then, use \((1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots\) to expand \((1 - \frac{2GM}{c^2r})^{-1/2}\) to linear order and \((1 - \frac{2GM}{c^2r})^{-1}\) to quadratic order in \(x = -\frac{2GM}{c^2r}\). This ensures that we retain terms through order \(1/c^2\) in both \((1 - \frac{2GM}{c^2r})^{-1/2}\) and \(c^2(1 - \frac{2GM}{c^2r})^{-1}\). Please leave \(e^2\) alone and do not replace it with an approximate expression until the end of part (b).

(b) You should find

\[
u_1 + u_2 = \frac{2GMe^2}{\ell^2 \left(1 - 4 \left(\frac{GM}{\ell c}\right)^2\right)} \quad \text{and} \quad u_1u_2 = -\frac{c^2(e^2 - 1)}{\ell^2 \left(1 - 4 \left(\frac{GM}{\ell c}\right)^2\right)}.\]

For completeness, please give the leading approximation to these expressions using the expansion of \(e^2\) in powers of \(E_{\text{Newt}}/mc^2\). Only \(u_1 + u_2\) will actually show up in evaluating the integral.

(c) To evaluate the second integral, define \(v = (u_1 - u)(u - u_2)\), and differentiate to obtain \(2udu = (u_1 + u_2)du - dv\). Thus,

\[
\frac{udu}{[(u_1 - u)(u - u_2)]^{1/2}} = \left(\frac{u_1 + u_2}{2}\right) \frac{du}{[(u_1 - u)(u - u_2)]^{1/2}} + \frac{1}{2} \frac{dv}{v^{1/2}},
\]

with \(v_1 = v_2 = 0\).

3. **Gravitational redshift in the PPN metric.** Hartle Problem 10.2. This problem and the next are applications of Chapter 9 to solar system tests of general relativity. Please read the two-page Section 10.2 if you have not already done so. The parametrized-Post-Newtonian framework (PPN) provides a means of organizing the predictions of different gravity theories. Hartle discusses the two most important parameters \(\gamma\) and \(\beta\). (See, e.g., Wikipedia for the others.) In general relativity, \(\beta = \gamma = 1\). However, in this problem you should keep \(\beta\) and \(\gamma\) arbitrary.

**Hint:** This problem tests our understanding of the derivation of gravitational redshift in Sec. 9.2. We are asked to work through the same steps as in Sec. 9.2, but for the PPN metric with arbitrary
β and γ rather than the Schwarzschild metric. To solve the problem, all that we need to know about the experiment of Vessot and Levine, is that it sets an upper bound on the deviation of the fractional redshift from the prediction of general relativity,

$$\left( \frac{\Delta \omega}{\omega} \right)_{\text{obs}} - \left( \frac{\Delta \omega}{\omega} \right)_{\text{GR}} = \left( \frac{1 - \omega_\infty}{\omega} \right)_{\text{obs}} - \left( 1 - \frac{\omega_\infty}{\omega} \right)_{\text{GR}} \lesssim 10^{-4},$$

where $\omega_\infty$ and $\omega_\ast$ are defined as in Sec. 9.2, and $\Delta \omega = \omega_\ast - \omega_\infty$.

Suppose that the real world is described by the PPN metric for some β and γ. Then, by substituting the PPN result for the observed quantities above, we obtain a constraint on β and γ. For the Earth, the relevant gravitational source for the Vessot-Levine experiment, the inside back cover of Hartle gives

$$\frac{GM_\oplus}{c^2} = 0.443 \text{ cm and } R_\oplus = 6.38 \times 10^8 \text{ cm.}$$

Using this information, please show that we obtain a constraint $|\gamma - \beta| \lesssim 10^5$. Is this interesting?

4. Deflection of light in the PPN metric. Hartle Problem 10.4. The goal of this problem is to work through the same steps that led to (9.63) and (9.64), and from (9.76) to (9.81), with the PPN metric (10.4) instead of the Schwarzschild metric.

Hint: To streamline the algebra in this problem, it is helpful to write the PPN metric in the equatorial plane as

$$ds^2 = -A(r)c^2dt^2 + B(r)er^2 + r^2d\phi^2 \quad \text{(equatorial plane } \theta = \pi/2)$$

where

$$A(r) = 1 - \frac{2GM}{c^2r} + \text{(order } 1/c^4) \quad \text{and } \quad B(r) = 1 + 2\gamma\left(\frac{GM}{c^2r}\right) + \text{(order } 1/c^4).$$

You should find

$$\Delta \phi = 2 \int_{r_1}^{\infty} \frac{B^{1/2}}{r^2} \left( \frac{1}{b^2 A} - \frac{1}{r^2} \right)^{-1/2},$$

where $b^2 = (\ell c/e)^2$. Your goal is then to show that this gives

$$\Delta \phi = 2 \int_0^{w_1} dw \left( 1 + \frac{\gamma GM}{c^2 b} w \right) \left( 1 + \frac{2GM}{c^2 b} w - w^2 \right)^{-1/2} + \text{(order } 1/c^4),$$

where $w = b/r$. This integral can be computed using an algebraic integration program such as Mathematica, and then expanded to first order in $GM/c^2 b$, however, you need not do this yourself. You may simply quote the result that integral (10) gives

$$\Delta \phi = \pi + \frac{1 + \gamma GM}{2} \frac{c^2 b}{2} + \text{(order } 1/c^4)$$

from which the PPN deflection angle is

$$\delta \phi_{\text{def}} = \Delta \phi - \pi \simeq \frac{1 + \gamma GM}{2} \frac{c^2 b}{2}.$$