Due: Fri 27 March 2015

Reading: Please continue to study Chapter 9, and read Chapter 10.

Problems (three pages total):

1. The radial equation in the Schwarzschild geometry. Last week, on Problem 1 (Hartle Problem 8.3), we derived the following three geodesic equations for motion in the equatorial plane $\theta = \pi/2$ of the Schwarzschild geometry:

\[
\frac{d}{d\tau} \left( \left(1 - \frac{2M}{r} \right) \frac{dt}{d\tau} \right) = 0, \\
\frac{d^2 r}{d\tau^2} + \left(1 - \frac{2M}{r} \right) M \left( \frac{dt}{d\tau} \right)^2 - \left(1 - \frac{2M}{r} \right) \left( \frac{d\phi}{d\tau} \right)^2 = 0, \\
\frac{d}{d\tau} \left( r^2 \frac{d\phi}{d\tau} \right) = 0.
\]

(1)

We will now perform a consistency check to confirm that these equations agree with our analysis in class of the radial motion of a particle moving in this geometry.

(a) Please interpret the first and third geodesic equation in terms of the Killing vectors $e$ and $\ell$ discussed in class and in Chapter 9. When expressed in terms of $e$ and $\ell$ instead of $dt/d\tau$ and $d\phi/d\tau$, show that the second equation becomes the purely radial equation

\[
\frac{d^2 r}{d\tau^2} + \left(1 - \frac{2M}{r} \right) \left( \frac{dr}{d\tau} \right)^2 - \left(1 - \frac{2M}{r} \right) \ell^2 r^3 = 0.
\]

(2)

(b) The quantity $\mathbf{u} \cdot \mathbf{u} = -1$ is conserved along geodesics by the definition of the 4-velocity. (This equation is just $g_{\alpha\beta} dx^\alpha dx^\beta = -d\tau^2$, divided by $d\tau^2$.) In class we showed that this equation can be written

\[
-\left(1 - \frac{2M}{r} \right) \left(1 - \frac{2M}{r} \right) \left( \frac{dr}{d\tau} \right)^2 + \frac{\ell^2}{r^2} = -1.
\]

(3)
Use this equation to further simplify Eq. (2) to
\[ \frac{d^2r}{d\tau^2} = -\frac{M}{r^2} + \frac{\ell^2}{r^3} - \frac{3M\ell^2}{r^4}. \]  
(4)

(c) Show that this is the same equation as
\[ \frac{d^2r}{d\tau^2} = -\frac{dV_{\text{eff}}(r)}{dr}, \]  
where \( V_{\text{eff}}(r) \) is the effective radial potential (effective potential energy per unit rest energy) derived in class and given by Hartle Eq. (9.28).

2. More on geodesics in Rindler space. Last week we worked hard to determine the relation \( X(T) \) for geodesics in Rindler space using two methods: (a) by deriving the geodesic equations (Hartle Eq. 8.14), eliminating \( d\tau \) and then integrating; and (b) by identifying a Killing vector \( \xi \) and then using the conservation of \( u \cdot u \) and \( u \cdot \xi \) along the geodesic.

Method (b) is the most widely applicable technique for determining geodesics in general relativity. Method (a) is usually the most computationally intensive, as we have now experienced firsthand.

This week, we will explore two more methods for solving this problem. The first is general. (We explored this approach in class on Thu 3/1 to study the of geodesics of 2D flat space in polar coordinates \((r, \phi)\).) The second is a one-liner, given that we already know the geodesics in polar coordinates in Euclidean space.

(a) The proper time \( \tau \) along a geodesic connecting two points \( A \) and \( B \) in spacetime is
\[ \tau_{AB} = \int_{\tau_A}^{\tau_B} d\tau, \]  
(6)
where in Rindler space,
\[ -d\tau^2 = ds^2 = -X^2dT^2 + dX^2. \]  
(7)
(Note that extremizing the proper time is the same as extremizing the action \( S = -mc^2 \int d\tau \).) Last week, we introduced an arbitrary parameter \( \sigma \) along the geodesic and than expressed Eq. (6) as an integral over \( \sigma \). This time, please instead write \( T = T(X) \) and express Eq. (6) as an integral
\[ \tau_{AB}[T(X)] = \int_{X_A}^{X_B} f(T, T', X) \, dX, \]  
(8)
where \( T = T(X) \) and \( T' = dT/dX \). You should find that \( f = f(T', X) \), with no explicit \( T \) dependence.
(b) What is the condition that the proper time be an extremum? Answer:

\[
\frac{X^2T'}{\sqrt{X^2T'^2-1}} = e, \tag{9}
\]

for some constant \(e\). Identify this conserved quantity with \(-\xi \cdot u = -g_{\alpha\beta} \xi^\alpha \frac{dx^\alpha}{dt}\) for the appropriate Killing vector \(\xi\). Hint: you’ll have to eliminate \(d\tau\) using Eq. (7).

(c) Solve Eq. (9) for \(T'(X)\) and then integrate to find \(T(X)\). Call the resulting integration constant \(T_0\). Hint: You’ll need the indefinite integral

\[
\int \frac{dX}{X\sqrt{e^2-X^2}} = \frac{1}{e} \cosh^{-1}\left(\frac{e}{X}\right). \tag{10}
\]

Finally, invert the result to show that

\[
X \cosh(T-T_0) = e = \text{const}. \tag{11}
\]

(d) Here’s the one-liner. In recitation section, we analyzed the metric

\[
ds^2 = dr^2 + r^2 d\phi^2 \tag{12}
\]

for 2D Euclidean space in polar coordinates, and showed that the geodesics are

\[
r \cos(\phi - \phi_0) = a = \text{const.} \tag{13}
\]

Since the metric (12) is related to the Rindler space metric (7) by the coordinate transformations \(r = X, \phi = -iT\), so are the geodesics. Show that gives Eq. (11) starting from Eq. (13). This sort of imaginary coordinate transformation of \(\phi\) is called analytic continuation or Wick rotation.

3. Gravitational “redshift” for massive particles. Hartle Problem 9.3. Just as a photon loses energy and momentum in escaping a massive star (cf. Sec. 9.2 in Hartle), so too does a massive particle.

Hint: Write \(e_0^\alpha = e_t^\alpha = (A,0,0,0)\) and \(e_1^\alpha = e_\phi^\alpha = (0,B,0,0)\) in Schwarzschild coordinates \((t,r,\theta,\phi)\), and determine \(A, B\) so that these are orthonormal basis vectors. The quantities \(E\) and \(P\) are components of the momentum 4-vector in the orthonormal basis: \(E = e_0 \cdot p\) and \(P = e_1 \cdot p\).

4. Proper time of descent toward a black hole. Hartle Problem 9.6. This is a short problem that provides an application of the effective potential \(V_{\text{eff}}(r)\).