Due: Fri 17 April 2015

Instructions: This is the second of three review problem sets in Physics 325. It will count for twice as much as a regular problem set. In addition to the formulae provided, you may consult the textbooks and your notes from lecture. However, in contrast to the regular weekly problem sets, you may not discuss this review problem set with anyone else. Please do all problems in Parts I and II, and choose one out of the two problems in Part III. Show all your work and indicate clearly which of the two problems you would like considered in Part III. This review problem set is due by the end of the day on Friday 17 April 2015.
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Useful formulae

Metric:
\[ ds^2 = g_{\alpha\beta}(x) \, dx^\alpha dx^\beta. \]

Proper time::
\[ d\tau^2 = -ds^2, \]
for a timelike worldline.

4-velocity:
\[ u^\alpha = \frac{dx^\alpha}{d\tau} \quad \text{(for a massive particle)}, \]
\[ u^\alpha = \frac{dx^\alpha}{d\lambda} \quad \text{(for light)}, \]
where \( \lambda \) is an affine parameter.

Dot product:
\[ \mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta}a^\alpha b^\beta. \]

Dot product of the four velocity with itself:
\[ \mathbf{u} \cdot \mathbf{u} = \begin{cases} -1 & \text{(for a massive particle)}, \\ 0 & \text{(for light)}. \end{cases} \]

Length, area, 3-volume and 4-volume for a diagonal metric:
\[ d\ell_1 = \sqrt{g_{11}} \, dx^1 \quad \text{(length in the } x^1 \text{ direction)}, \]
\[ dA_{12} = \sqrt{g_{11}g_{22}} \, dx^1 dx^2 \quad \text{(area in the } x^1 x^2 \text{ directions)}, \]
\[ dV = \sqrt{g_{11}g_{22}g_{33}} \, dx^1 dx^2 dx^3 \quad \text{(3-volume in the } x^1 x^2 x^3 \text{ directions)}, \]
\[ dV = \sqrt{-g_{00}g_{11}g_{22}g_{33}} \, dx^0 dx^1 dx^2 dx^3 \quad \text{(4-volume)}, \]

Schwarzschild metric:
\[ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]
Effective radial potential for a massive particle in the Schwarzschild metric:

\[ V_{\text{eff}}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}. \]

Energy conservation:

\[ \mathcal{E} = \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r). \]

critical points of the effective radial potential:

\[ \frac{dV_{\text{eff}}(r)}{dr} = 0 \quad \text{at} \quad r = \frac{\ell^2}{2M} \left[ 1 \pm \sqrt{1 - \frac{1}{12} \left( \frac{M}{\ell} \right)^2} \right]. \]

Killing vectors and conserved quantities: The metric \( ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta \) has a Killing vector \( \eta \) if the transformation \( x^\alpha \rightarrow x^\alpha + \epsilon \eta^\alpha \) is a symmetry of the metric for constant \( \epsilon \). In this case, the quantity

\[ K = \eta \cdot u \]

is conserved along geodesics. Here \( u \) is the 4-velocity for timelike or null geodesics, and \( u^\alpha = dx^\alpha / ds \) for spacelike geodesics.

Functional derivatives and Euler-Lagrange equations:

The functional

\[ F[x^\alpha(\sigma)] = \int_{\sigma}^{\sigma_B} f(x^\alpha, \dot{x}^\alpha, \sigma) d\sigma \]

is extremized when its functional derivatives with respect to the functions \( x^\alpha(\sigma) \) all vanish,

\[ 0 = \frac{\delta F}{\delta x^\alpha(\sigma)} = \frac{\partial f}{\partial x^\alpha} - \frac{d}{d\sigma} \left( \frac{\partial f}{\partial \dot{x}^\alpha} \right), \]

where \( \dot{x}^\alpha = dx^\alpha / d\sigma \). These equations are known as the Euler-Lagrange equations.

Geodesic equation:

The geodesic equation is

\[ \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0, \]

where the \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols.
Part I. Shorter problems. Please do Problems i, ii, and iii, and optionally Problem iv for extra credit.

i. 4-velocity of an observer (5 pts). Please derive an expression for the 4-velocity of an observer at fixed radius \( r = R \), and fixed \( \theta, \phi \), in the Schwarzschild geometry. Express your answer by giving the components \( u^\alpha \) in the Schwarzschild coordinate basis.

ii. Uniformly accelerating universe (3 pts). What is meant by a uniformly accelerating universe and what is its relation to the worldline of a uniformly accelerating particle?

iii. Effective radial potential (7 pts). Please write down the effective radial potential \( V_{\text{eff}}(r) \) for a massive particle moving in the Schwarzschild geometry. Briefly comment on each of the three terms. Then sketch \( V_{\text{eff}}(r) \) in the case \( \ell/M > 4 \). Draw horizontal lines indicating the energy \( \mathcal{E}_i \) for: \( \mathcal{E}_1 \) a stable circular orbit, \( \mathcal{E}_2 \) a bound orbit, \( \mathcal{E}_3 \) an unbound orbit with nonzero minimum radius, \( \mathcal{E}_4 \) an unstable circular orbit, and \( \mathcal{E}_5 \) an orbit that starts at \( r = \infty \) and makes it all the way to \( r = 0 \). Label turning points with bold dots. How does the plot of the effective potential differ in the \( \ell/M < \sqrt{3/2} \) case? In the Newtonian case?

iv. Penrose diagram (extra credit). What is a Penrose diagram? Draw your favorite Penrose diagram and label the various infinities.

Part II. Problem.

1. Length and area in the wormhole metric (10 pts).\(^1\) Consider the wormhole metric

\[
 ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2).
\]

Here, the ranges of \( t, \theta \) and \( \phi \) are the usual ones, but \( -\infty \leq r \leq \infty \). (Note that this differs from the range \( r \geq 0 \) of spherical coordinates.)

(a) Verify that the length of a curve from \( r = 0 \) to \( r = R \) at constant \( t, \theta, \phi \) is simply \( R \).

(b) Calculate the three dimensional volume on a \( t = \) constant slice of the wormhole geometry bounded by two spheres of radius \( R \) on each side of the throat (i.e, bounded by \( r = -R \) and \( r = R \)).

\(^1\)This is Problem 7.17 in Hartle.
Part III. Problems. Please do one of Problems 2 and 3.

2. Deflection of light in another gravity theory (15 pts)

Suppose in another theory of gravity (not Einstein’s general relativity) the metric outside a spherical star is given by

\[ ds^2 = \left(1 - \frac{2M}{r}\right)[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \] (2)

In this problem, we will calculate the deflection of light by a spherical star in this theory assuming that photons move along null geodesics in this geometry. As in our analysis of orbits in class, please restrict to the equatorial plane \( \theta = \pi/2 \).

(a) For light, we define the 4-velocity to be \( u^\alpha = dx^\alpha/d\lambda \), where \( \lambda \) is an affine parameter and \( x^\alpha = (t, r, \theta, \phi) \). Please write out the condition \( u \cdot u = 0 \) in terms of the explicit components of the metric and of \( u \).

(b) Identify two symmetries of the metric. What are the corresponding Killing vectors \( \xi \) (timelike) and \( \eta \) (spacelike), and conserved quantities \( e = -\xi \cdot u \) and \( \ell = \eta \cdot u \)?

(c) Eliminate \( dt/d\lambda \) and \( d\phi/d\lambda \) from the result of part (a) in favor of the constants \( e \) and \( \ell \) found in part (b). Then, solve for \( dr/d\lambda \) to show that

\[ \frac{1}{\ell} \frac{dr}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{1/2}, \] (3)

where \( b = |\ell/e| \).

(d) Finally, by substituting the definition of \( \ell \) into the left hand side, obtain an expression for \( d\phi/dr \). The integral of this expression gives the change \( \Delta \phi \) for the orbit.

(e) (extra credit) In flat space, we have \( \Delta \phi = \pi \) (no deflection) and

\[ \frac{d\phi}{dr} = \frac{1}{r^2} \left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-1/2}. \] (4)

How does the result obtained in part (d) compare to the flat space result? Can you think of a simple reason why?

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\(^2\)This is Problem 9.18 in Hartle with a few more hints.
3. Geodesics in de Sitter space (15 pts).

The metric for 1 + 1 dimensional de Sitter space in Friedman-Robertson-Walker form is

\[ ds^2 = -dt^2 + a^2(t)dx^2, \]

where \( a(t) = e^{t/R} \).

(a–f) (extra credit) For extra credit, please work through this problem for arbitrary \( a(t) \) rather than the \( e^{t/R} \) of de Sitter space.

(a) Please write an integral expression for the proper time along a timelike worldline

\[ t = t(\sigma), \quad x = x(\sigma), \]

connecting two points \( A \) and \( B \) in de Sitter space. Here \( \sigma \) is an arbitrary parameter along the worldline. Express your answer as a functional

\[ \tau_{AB}[t(\sigma), x(\sigma)] = \int_{\sigma_A}^{\sigma_B} f(t, x, \dot{t}, \dot{x}) d\sigma, \]

where a dot denotes \( d/d\sigma \). Hint: You should find that \( f = f(t, \dot{t}, \dot{x}) \), with no explicit dependence on \( x \).

(b) By extremizing the functional found in part (a), and then re-expressing your result in terms of \( d/d\tau \) derivatives rather than \( d/d\sigma \) derivatives, obtain the geodesic equations for \( t \) and \( x \).

(c) From the geodesic equations, please read off the nonvanishing Christoffel symbols.

(d) Identify a symmetry of this metric. What is the corresponding Killing vector \( \eta \) and conserved quantity \( K \)? Looking back through your steps in part (b), identify one of the geodesic equations as the statement that \( K \) is conserved.

(e) Show that \( x = \text{constant} \) is a solution to the geodesic equations.

(f) Consider two particles, one at \( x = x_1 \) and the other at \( x = x_2 > x_1 \). What is the distance \( s_{12} \) between the two particles at time \( t \)?