are for your benefit only; thus no record will be kept of the grades.

Special reports or projects:
If there are any topics that you and/or your group feels that they want to explore in greater detail you may research the topic and write a special report on it, or set up an experiment, or suggest an experiment, or anything else that would be relevant to the material. Please feel free to consult me, any other professor, any other physics student, or any other resource that you can think of for references, help, etc.

Amendments:
These rules may be amended during the semester with respect to details but not in essence.

6. For more details and results write to the author.
7. The author wishes to thank Dr. Robert Karplus of the University of California at Berkeley for this observation.
9. C. Rogers, Ref. 4, pp. 23 and 137.
10. Adopted from a sheet given to the author by Dr. Clyde Bratton, Cleveland State University.
11. This is the handout sheet as it evolved using the experience of the first two semester courses described.

Experiment to Measure the Increase in Electron Mass with Velocity

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An experiment is described for measuring the mass increase with velocity of relativistic electrons. A beta-ray spectrometer is used to momentum-analyze electrons emitted from a TF source, and a semiconductor detector is used to measure the energy of the electrons. The energy calibration of the semiconductor detector is established absolutely. The experimental results serve to verify the predictions of special relativity, and provide precise values for the electron rest mass and the speed of light.

INTRODUCTION

It is becoming common practice to introduce some basic concepts in special relativity early in the undergraduate physics curriculum. However, many of the basic concepts presented in the classroom have for the most part been avoided in the undergraduate laboratory. This omission is largely due to the complexity and cost of equipment needed to observe and record relativistic effects. The filmed experiment reported by Bertozzi depicts an experiment that measures the speed of electrons in the MeV range using time-of-flight techniques. Electrons with kinetic energies in the range 0.5 to 15 MeV are produced with a linear accelerator.

Equipment designed to measure the mass increase with velocity for relativistic electrons is commercially available and suitable for student use. These experiments are based on the classic work of Zahn and Spees and involve either separate or simultaneous measurements of the velocity and momentum of energetic electrons from a beta-ray source in uniform electric and magnetic fields.

In this report, we describe an experiment for observing the mass increase with velocity of
relativistic electrons by measuring momentum and kinetic energy. A beta-ray spectrometer is used to momentum-analyze the electrons emitted from a \(\text{Tm}^{239}\) source, and a semiconductor detector is used to measure the energy of the electrons. The semiconductor detector, which is placed at the focus of the 180° beta-ray spectrometer, is calibrated absolutely to better than 2% using standard techniques. The experimental results not only serve to verify the predictions of special relativity, but also provide the student with rather precise values for the electron rest mass and the speed of light.

The experiment as presented in this report is part of an advanced physics lab which is required of all our physics majors in their junior year. Data collection time is about 15 to 20 hours, depending on the number of energy points selected. Unless students have some familiarity with the equipment, especially with nuclear detection techniques, additional time must be allotted for equipment familiarization. Therefore, at least 30 hours should be allotted for students to carry out this experiment to completion satisfactorily. Of course, the experiment may be completed in a much shorter time depending on the amount of supervision given to the students, and on the number of energy points measured.

**THEORY**

Classically, the momentum, \(p\), of a particle and its kinetic energy, \(T\), are related by the expression

\[
p^2 / 2T = m, \tag{1}
\]

where \(m\) is assumed to be a constant parameter characteristic of the particle. However, for particles moving with velocities near the speed of light, \(c\), the special theory of relativity is applicable and the relation connecting the total energy, \(E\), of the particle and its momentum, \(p\), is given by

\[
E^2 = p^2c^2 + m_0^2c^4, \tag{2}
\]

where \(m_0\) is the rest mass of the particle. The total energy is expressed by

\[
E = T + m_0c^2. \tag{3}
\]

Combining Eqs. (2) and (3) yields the relativistic analog of Eq. (1),

\[
p^2 / 2T = m_0 + (1/2c^2)T. \tag{4}
\]

Special relativity, therefore, predicts that \(p^2 / 2T\) is a linear function of the kinetic energy with a slope \((2c^2)^{-1}\) and an intercept of \(m_0\).

Thus, experimental verification of Eq. (4) requires a determination of both the particle's energy and its momentum. For an electron of charge \(e\) moving in a uniform magnetic field, \(B\), which is normal to the plane of motion, the covariant expression for the momentum, neglecting radiative effects, is

\[
p = eBR, \tag{5}
\]

where \(R\) is the radius of the electron's path. If a beta-ray source with a continuous spectrum of energies up to some end-point energy is placed in the magnetic field, then electrons with different momenta may be selected by varying \(B\) for the fixed radius of curvature \(R\). The kinetic energy of these electrons can then be determined with the semiconductor detector suitably located at the focus of the beta-ray spectrometer. Substituting Eq. (5) into Eq. (4) we then get

\[
(eBR)^2 / 2T = m_0 + (1/2c^2)T, \tag{6}
\]

where the quantities on the left side of Eq. (6), \(B, R,\) and \(T\), are measured directly and \(e\) is the fundamental electron charge.

**EXPERIMENTAL DETAILS**

The beta-ray spectrometer shown in Fig. 1 is of the semicircular type having a first-order focus for 180° deflection. The 50 µc \(\text{Tm}^{239}\) \(\beta\) source (B) and the semiconductor detector (A) are mounted

**Fig. 1. Diagram of beta-ray spectrometer chamber. See text for identification of components.**
4 in. apart on a removable bar (D). The 765 keV electrons available from the $^{208}$Tl source (half-life: 3.75 yr) is selected because of the limited stopping power of the detector and shielding considerations. The semiconductor detector is a Si(Li) counter with 3-mm depletion depth and 30-mm$^2$ sensitive area.

Four sets of slits (C) are provided to define the radius of curvature of the central ray at $R=2$ in. and to adjust the resolution and transmission of the spectrometer. A set of slits immediately in front of the $\beta$ source (not shown in the figure) defines the active width of the source, and is usually set 1 mm apart. The intermediate slits are set 2 mm apart and select the pencil of rays from the source which come to a focus at the exit slit immediately in front of the detector. The plate (E) shields the detector from the source.

An O-ring seal is made between the spectrometer chamber and a cover plate. A mechanical pump is used to obtain a working pressure of about 25 $\mu$, which is recorded with the thermocouple gauge (F) on the evacuation line. A hole fitted with a Teflon collar (G) is provided in the side of the chamber for measuring the magnetic field at the central radius.

A complete diagram of the experimental system is shown in Fig. 2. The magnet, model AL7500M available from Alpha Scientific, is modified to accommodate 8-in.-diam pole faces at a pole gap separation of 2 in. The magnetic field is observed to be uniform to within 0.1% over about a 3-in. radius. Since the magnitude of the magnetic field, $B$, appears squared in Eq. (6), it is essential that it be determined accurately if Eq. (6) is to be verified with high precision. Therefore, the field is measured to 0.5% using a rotating coil gaussmeter whose calibration is checked against a 1000-gauss standard magnet.

Two different methods are applied to establish an energy calibration for the charged particle spectrometer, pulse height versus energy. The simplest and most direct method of obtaining an energy calibration is to use an internal conversion electron source, say $^{207}$Bi, which has discrete electron groups of well-known energies. This procedure, however, is of relative significance since the energies assigned to the conversion electrons are obtained by using Eq. (6), which is to be verified. This procedure is very useful though, because it is most direct and does demonstrate the linear variation in the quantity $p^2/2T$ with energy.

The second and most appealing method is to establish an absolute calibration for the charged particle spectrometer. If $T$ is the energy deposited in the sensitive volume of the detector, then the charge released is given by relation

$$Q = eT/\epsilon,$$

where $\epsilon$ is the average energy required to produce

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an electron-hole pair in the semiconductor and depends on the detector material, temperature, and type of radiation incident. For $\beta$ particles on silicon at 300°C, Pehl et al. report a value: $\epsilon = 3.67 \pm 0.02$ eV/e-h pair. The absolute charge calibration of the pulse height analyzing system is obtained by applying a charge of known magnitude at the detector input and observing the corresponding pulse height in terms of channel number. Charge is injected in the usual way through the series capacitor, $C_T$, of the charge terminator (see Fig. 2) provided with the linear voltage pulser. The energy corresponding to a voltage pulse, $V$, is then given by

$$T = C_T V \epsilon/e.$$  \hspace{1cm} (8)

Since $\epsilon$ is known to about 0.6%, the voltage pulse, $V$, and $C_T$ must be measured to a corresponding degree of accuracy. The pulser signal is easily measured with a digital voltmeter. However, $C_T$, which is usually of the order of 2 pF, is not easily measured to the required degree of accuracy on capacitance bridges generally available in an undergraduate laboratory. This measurement can be made by the student using the relatively simple and inexpensive analog capacitance bridge described in the appendix.

**EXPERIMENTAL RESULTS**

Typical pulse height spectra are shown in Fig. 3 for several values of electron momenta. Data collection times average about 1 h per point with shorter running times realized for the lower energies, and longer running times needed near the end point of the $^{214}$Pb beta-decay spectrum. In general, the peaks are sufficiently well defined so that their location can be determined with reasonable accuracy by visual inspection. In practice, the range of electron energies which provide usable spectra within a reasonable collection time is from 150 to 700 keV. The limitation at low energies is imposed by detector noise.

The results of two experimental runs are shown in Fig. 4, where $p^2/2T$ is plotted against the kinetic energy $T$. The circle points are obtained using the relative energy calibration for the semiconductor detector, while the triangle points are based on an absolute calibration. Both sets of data agree very well with the theoretical prediction indicated by the solid curve. The error flags on the data points reflect the uncertainty in the measurement of $B$ (0.5%), $R$ (1%), and the energy calibration resulting from a 1% error in the determination of $C_T$ and a 0.6% error reported for $\epsilon$. The error in peak identification (see Fig. 3) is about 5 keV. A least-squares fit to all the data points in Fig. 4 yields $8.99 \pm 0.30 \times 10^{-31}$ kg for

![Fig. 4. Experimental results for the quantity $p^2/2T$ vs $T$. The circle points are based on a relative energy calibration for the semiconductor counter, while the triangle points are based on an absolute calibration. The solid curve is the theoretical prediction. The ordinates on the right-hand-axis express mass in terms of Energy/$c^2$ with energy in MeV. For $m$ in grams and $c$ in cm/sec, $E/c^2 = 0.625 \times 10^9 m$ (MeV sec²/cm²).](image)
$m_0$ and 2.99±0.11×10^5 m/sec for $c$ in excellent agreement with the accepted values. A least-squares fit of only those data points based on the absolute spectrometer calibration yields 8.92±0.49×10^{-21} kg for $m_0$, and 2.96±0.19×10^5 m/sec for $c$.

ACKNOWLEDGMENT

The authors would like to thank Mr. Tom Coyle for constructing the spectrometer chamber.

APPENDIX: ANALOG CAPACITANCE BRIDGE

The circuit diagram for the analog capacitance bridge is shown in Fig. 5. The bridge consists of a differentiator, integrator, and comparator. The terminator capacitor (unknown) is at the input of the differentiating circuit. A signal $v_0 \sin \omega t$ applied to this input produces an output signal $-\omega R_3 C_3 v_0 \cos \omega t$. The effective capacitance, due to the connectors, $c_i$, and the operational amplifier input, $c_{in}$, ($c_{in}+c_i\approx 6$ pF), is minimized since it appears across the low effective input impedance of the operational amplifier. The input signal to the differentiator is also applied to an integrating circuit whose output is $(1/\omega R_2 C_1) v_0 \cos \omega t$. The two outputs are then fed to a dual trace oscilloscope or an ac null detector and the frequency of the signal is varied until the amplitudes of the differentiated and integrated signals are equal. The value for the terminator capacitance is then

$$C_T = (\omega^2 R_1 R_2 C_1)^{-1}.$$

The values of the components $R_1$, $R_2$, and $C_1$ are chosen so that the bridge will balance at about 30 kHz for $C_T \approx 2$ pF. The values selected for these elements are sufficiently large so that they may be easily measured on a standard 0.1% bridge. In practice, the oscillator frequency is resettable to about 0.1% and is measured to this degree of accuracy by pulse counting techniques. The overall accuracy of the bridge, considering the frequency roll off of the operational amplifiers, is about 1% up to an operating frequency of 40 kHz. For proper operation of the bridge, it is essential that the oscillator output be undistorted and free of both high and low frequency noise. High-frequency noise causes distortion at the
output of the differentiator, while low-frequency noise may cause saturation of the integrator. Because of the small value of $C_T$, the input connections to the charge terminator must be shielded from the output connections. $C_2$ is chosen to be the minimum capacitance which will maintain circuit stability at the maximum expected value of $C_T$.

5. The value for $\epsilon$ reported in Ref. 4 is not an absolute determination. It is based on an observation of the semiconductor response to gamma rays and internal conversion electrons whose energies are well established by spectroscopic techniques which use the Einstein relation. However, an absolute determination of $\epsilon$ is possible, in principle at least, by using electrons from a low-energy accelerator. It is in this sense that we distinguish this second measurement as an absolute one, since the measurement of $p^2/2T$ would then involve the measurement of distance, charge, voltage, and magnetic field. We also note that the formation energy for the semiconductor material is an intrinsic property of the material for the given ionizing particle and it is subject to theoretical analysis.

A Physicist's Introduction to Bayesian Statistics. II

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This sequel to an earlier paper applies Bayesian estimation theory to several probability distributions of importance in physics [Poisson, Maxwell, Rayleigh, exponential, Druyvesteyn, and Gauss (zero mean) distributions]. A single tractable and physically interesting loss function (the square of the relative estimation error) is utilized. Bayes estimates are obtained by using a single conjugate prior distribution (the gamma distribution) and the vague prior limit is taken. The estimates obtained tend to the mean values of the respective posterior distributions for large sample sizes. A practical aspect of the theory is demonstrated by obtaining an optimal estimate of the most probable speed for a Maxwell distribution of speeds.

Optimal parameter estimation using Bayesian statistics was recently reviewed in this Journal by the present author as part of an introduction to Bayesian statistics for physicists. Readers are requested to consult this earlier paper for definitions and basic concepts. The present paper supplements the earlier one by demonstrating procedures for applying the theory. In the Bayesian framework of parameter estimation a loss function is defined (that is, a function which describes the consequences of an estimation error) and an optimal (or Bayes) estimate is an estimate which yields a minimum mean loss. If $L(w, \tilde{w})$ denotes a loss function where $\tilde{w}$ is the true value of a parameter and $\tilde{w}$ an estimate of that parameter, then $\tilde{w}^*$ is said to be a Bayesian estimate if

$$E[L(W, \tilde{w}^*)] = \inf_{\tilde{w}} E[L(W, \tilde{w})],$$

where the expectation is with respect to the posterior distribution of $W$. (In this paper a quantity thought of as a random variable will be denoted by a capital letter, the lower case of the letter being reserved for a realization or fixed value of that random variable.)

The number of tractable loss functions is small. The two most widely used tractable loss functions, $L_1(w, \tilde{w}) = a_1 \mid w - \tilde{w} \mid$ and $L_2(w, \tilde{w}) = a_2(w - \tilde{w})^2$,