Fourier Optics

In this experiment you will explore the Fourier transform using an optical system. Let’s begin with a brief review. What is a Fourier transform? Formally a Fourier transform pair is defined as,

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \, d\omega, \]

\[ g(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt. \]

When you see \( e^{-i\omega t} \) you should be thinking of sine waves (recall \( e^{-i\omega t} = \cos \omega t - i \sin \omega t \)). When you see \( \int \) you should be tinkling of a continuous sum, in this case over frequencies \( \omega \). So, in the first equation, we are representing the time dependent function \( f(t) \) as a continuous sum of sine waves, where the function \( g(\omega) \) tells us how much of each sine wave is needed in the sum. A Fourier transform basically tells us what frequencies are necessary to produce a given signal. The second equation above tells us how to find the weighting function \( g(\omega) \) for a given time dependent signal \( f(t) \).

The above description considers a time dependent signal, but in this experiment we will be interested in a signal that varies spatially. In this case we can write the Fourier transform pair,

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx} \, dk, \]

\[ g(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx, \]

where \( k \) is a spatial frequency. To simplify the notation let’s use \( \mathcal{F} \) to denote the Fourier transform and \( \mathcal{F}^{-1} \) to denote the inverse Fourier transform. The we have,

\[ f(x) = \mathcal{F}^{-1}[g(k)], \]

\[ g(k) = \mathcal{F}[f(x)]. \]
Before going any farther, let’s make sure we have some of the common Fourier transform pairs in our head. Find the Fourier transform of the following signal,

\[ \text{rect}(x) = \begin{cases} 1/a, & |x| < a/2, \\ 0, & |x| > a/2. \end{cases} \]

Sketch both \( \text{rect}(x) \) and its Fourier transform \( \mathcal{F}[\text{rect}(x)] \). How are the widths of these two functions related? As \( a \) gets very large, what happens to \( g(k) \)?

We can use \( \text{rect}(x) \) to define another important function, the delta function. Sketch some arbitrary function \( h(x) \). On the same plot sketch \( \text{rect}(x) \). Now consider the integral

\[ \int_{-\infty}^{\infty} \text{rect}(x) h(x) \, dx. \]

Where is this integral on the sketch you just made? What happens to the integral as you let \( a \to 0 \)? It is with this limit that we define the delta function

\[ \delta(x) = \lim_{a \to 0} \text{rect}(x). \]

Then we have,

\[ \int_{-\infty}^{\infty} \delta(x) h(x) \, dx = h(0), \]

or more generally,

\[ \int_{-\infty}^{\infty} \delta(x - b) h(x) \, dx = h(b). \]

Now let’s get back to Fourier transforms. What is \( \mathcal{F}[\delta(x)] \)? What is the Fourier transform of \( \cos(2\pi x/\Lambda) \) (Hint: \( \delta(x) \) and 1 are a Fourier transform pair)?

So, what does all of this have to do with lenses and optics? It turns out that a lens can be used to take the spatial Fourier transform of an object. We won’t develop the theory behind this, but if you interested in learning more, have a look at [http://atomoptics.uoregon.edu/~dsteck/teaching/optics/](http://atomoptics.uoregon.edu/~dsteck/teaching/optics/).

The experiment

Begin by setting up the coherent optical processor shown in Fig. ???. This processor illuminates an object (a two dimensional picture or mask of some
sort) in plane P1 with a collimated beam of laser light. Lens L1 then takes a Fourier transform of the object, which appears in plane P2, the Fourier plane. Finally, lens L2 takes an inverse Fourier transform to recreate an image of the original object. The reason that this is called an optical processor is that you can manipulate the image by placing a mask in the Fourier plane, that is, by manipulating the frequencies that make up the original picture.

The laser that you will use has a “spatial filter” on the front. Please do not adjust this. A spatial filter is a kind of optical processor that eliminates high frequency noise on the laser beam. So, what you see coming out of this laser is a nice uniform diverging beam. The first thing you need to do is collimate this beam. Place a lens in front of the beam so that the beam that emerges maintains a constant diameter for the entire length of the room. Now insert lenses L1 and L2 with the appropriate spacing between them.

Let’s begin by convincing ourselves that what appears in the Fourier plane is in fact the Fourier transform of the object. We’ll do this with some simple objects. The first is nothing. That’s right nothing, just a beam of roughly uniform brightness. The Fourier transform of this uniform beam should be a delta function. Stick a piece of paper in the Fourier plane. Do you see the delta function?

Now let’s move on to something a bit more complicated, a rect(x) function, which is just a long narrow slit. You can find a mask on the table with several slits of various widths. Does the Fourier transform have the shape you expect? How does the pattern you see in the Fourier plane depend on
the slit width? Does this make sense?

We don’t have a mask that is exactly sinusoidal, but something similar is a set of many parallel slits. Place this mask in the beam and observe the Fourier transform. For a cosine, you would expect two delta functions. What do you see for the slits? Does this make sense? think about the smooth variations in the sine function compared to the sharp edges of the slits.

Next have a look at the Fourier transform of a simple grid. In this case the one dimensional transform given above is insufficient. Here we need the two dimensional Fourier Transform,

\[ f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; g(k_x, k_y) e^{i(k_xx + k_y y)}, \]

\[ g(k_x, k_y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; f(x, y) e^{-i(k_xx + k_y y)}. \]

Start with a fine grid so that you can clearly see the Fourier transform and then switch to a coarse grid and have a look in the image plane.

Now is a good time to start thinking about filtering the signal in the Fourier plane. There are two kinds of filters that are easy to make, a thin wire and a thin slit. Experiment with filtering in the Fourier plane and observe the results in the image plane. Can you think of a filter that would allow you to remove the vertical lines, but keep the horizontal lines in the image of the grid?

Cut an odd shaped hole in a piece of aluminum foil. Design a filter to “find” the edges of the hole.

Try a low contrast object, a fingerprint on a glass slide, and see if you can design a filter to enhance the contrast.

If you want to record the Fourier transforms and images you are creating, there is a camera available, just place it directly in the plane where you want to observe the intensity pattern. This camera is connected to the computer where you can store the images. The camera is very sensitive to light so we have placed a filter on the front to severely attenuated the beam. This filter is labeled ND=3.0, which corresponds to an attenuation of 1000x. Please do not remove the filter or you may damage the camera. In fact, you may need additional filters to keep from saturating the camera, particularly when looking in the Fourier plane. Additional filters can be screwed on the the front of the ND=3.0 filter, but never remove the ND=3.0 filter.