Part I: Due Wednesday, Sept 21 by 5pm. Part II is due Friday Sept 23 by 5pm. Put your work into the appropriate folder in the bin outside Professor Donnay’s office or hand the homework in during class.

Part I
S1. In class we proved the following. Theorem: Let A and B be sets with $A \subseteq B$. If B is countable then A is countable.

a. State the converse of this theorem. Is the converse true (give a proof) or false (give a counter-example).

b. State the contraposition of this theorem. Is the contraposition true or false?

c. Use the result of (b) to give an example of an uncountable set. You may assume that the closed unit interval $I = [0, 1]$ is uncountable (we proved that in class). Try to make up a set that is not exactly the same as the ones we have seen in class already.

S2. Equivalence Relation. a. For the usual relation $\leq$ between real numbers (ex. $2 \leq 3$, $-4 \leq 1$), determine whether $\leq$ satisfies the three required properties of an equivalence relation. For each property, decide whether $\leq$ satisfies it or not. Just use your common sense knowledge about numbers; no formal proof required.

b. Real World Connection: Give an example from the everyday world (i.e. non-mathematical) of a notion of “sameness”. For example, “two human beings are the same if they have the same gender”. Then check whether your notion of sameness satisfies the three properties of equivalence relation or not.

S3. Finish the following definition: A sequence $s_n$ converges to a limit $L$ if 

Study the definition but then close your books and try to write out the definition from memory. You will be asked for the definition on this week’s quiz.

Morgan, Ch 3, #1, 4, 5, 6 (do not need to do rigorous proofs).

Morgan, Ch 3, #10 Use your calculator (or Excel) to generate a data table. Make a conjecture as to what the limit seems to be. Then practice using tolerance value and waiting time.

a. For $\epsilon = .3$, determine from your data the waiting time $N$. Draw both the two dimensional and one dimensional diagrams to illustrate this result.

b. For $\epsilon = .05$ determine the waiting time $N$ from your data.
Hand in a copy of your data (i.e. all the values of n and sₙ that you generated to help answer this question). Go up to s₂₅ and list your values to three decimal places.

S3. Predict the limit L of the sequence sₙ = 2 + \( \frac{1}{n^2} \). Determine N when \( \epsilon = .1 \). Do this via algebra rather than using a calculator.

Part II
S1. Find an example of a fractal shape or some other type of interesting mathematical shape on the web. Post the web link and a brief description of the example in the course Blackboard site. Add your entry to the Discussion Board: Fractal. See my entry as an example.

S2. Equivalence relation. Show that 'same cardinality' satisfies transitivity, the third property of equivalence relation. This will finish the proof that cardinality is an equivalence relation between sets. Under the "what to do when you do not know what to do" - start out by drawing a picture that illustrates the general situation described: sets A, B, C and functions between various of the sets. Remind yourself of the relevant definitions by re-writing them: B has the same cardinality as A, which we write as \( A \approx B \) if there exists a one-to-one and onto function \( f : A \rightarrow B \).

S2. Give examples of the following types of sequences. Give the formula for \( sₙ \). Draw a two dimensional picture of the sequence.
   a. Increasing, bounded, that converges to -3.
   b. Oscillating, bounded that converges to 4.
   c. Oscillating, bounded that diverges.
   d. Increasing, unbounded that diverges.
   e. Decreasing, bounded, that converges to 3.

S3. a. Predict the limit L of the sequence \( sₙ = 2 + \frac{1}{n^2} \). Then use the formal \( \epsilon - N \) definition of limit to give a rigorous proof of the limit.
   b. For the sequence \( sₙ = 3 + 2^{-n} \), decide what the limit is. Then use the \( \epsilon - N \) definition of limit to give a rigorous proof of the limit.

S4. (Note: this problem does not have to be done "rigorously". We will return to these types of examples a bit later in the semester. This is a warm up exercise).
   a. Define the sets \( Sₙ = [\frac{1}{n}, 1 - \frac{1}{n}] \) for \( n \geq 2 \). Determine \( \bigcup_{n=2}^{\infty} Sₙ \). Draw some pictures to help you understand the sets.
   b. Define the sets \( Tₙ = (1 - \frac{1}{n}, 1 + \frac{1}{n}) \) for \( n \geq 2 \). Determine \( \bigcap_{n=2}^{\infty} Tₙ \).
HW wk 3: Math 301

Part 1: S1. Theorem: Let $A, B$ be sets with $A \subseteq B$

If $B$ is countable then $A$ is countable.

Converse: If $A$ is countable then $B$ is countable.

Counter-example: Let $A = \mathbb{N}$ and $B = \mathbb{R}$

$A \subseteq B$, $A$ is countable yet $B$ is uncountable.

\( \square \) Contraposition: If $A$ is not countable then $B$ is not countable.

Let $A, B$ be sets with $A \subseteq B$.

i.e. \( \{ \) If $A$ is uncountable then $B$ is uncountable.

\( \square \) \( \square \) I = [0, 1] is uncountable. Let $I > A$.

Let $B = (-2, 8) = a$ new interval.

Then $B > A$ so $B$ is uncountable.
5.2. Equivalence Relation

3. Is "≤" an equivalence relation?

(P1) x ≤ x for all x ∈ R

(P2) if x ≤ y, i.e. x ≤ y
    then y ≤ x,
    i.e. y ≤ x.

This can be false: x = 2, y = 3
Then x ≤ y but y is not ≤ x.

(P3) Transitivity holds
    if x ≤ y and y ≤ z
    then x ≤ z.

6. real world example that satisfies all of sameness. Does it satisfy all 3 properties?
3. **Defn:** A sequence \( s_n \) converges to a limit \( L \) if given \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) \) such that if \( n > N \), then \( |s_n - L| < \varepsilon \).

Ch 3) 1. \( 1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \ldots \), 0, \( \frac{1}{2^n} \), ... approaches 0.

4. \( a_n = \frac{1 + (-1)^n}{n} \)
\( a_1 = 0, a_2 = \frac{2}{2}, a_3 = 0, a_4 = \frac{2}{4}, \)
\( a_5 = 0, a_6 = \frac{2}{6}, \ldots, \)
\( \frac{0}{n} \) if \( n \) odd
\( \frac{2}{n} \) if \( n \) even
\( a_n \to 0. \)

5. \( a_n = (-1)^n \left( 1 - \frac{1}{n} \right) \)
\( a_1 = 0, a_2 = \frac{1}{2}, a_3 = -\frac{2}{3}, a_4 = \frac{3}{4}, a_5 = -\frac{4}{5}, \)
\( a_6 = 0 + \frac{5}{6}, a_7 = -\frac{6}{7}, \ldots \)
Sequence does not have a limit.
Values approach both +1 and -1.
So does not approach a single value.
Diverges.

6. \( a_n = 1 + (-1)^n \)
\( a_1 = 0, a_2 = 2, a_3 = 0, a_4 = 2, \ldots \)
Diverges. Oscillates.
Ch 5 #10. \( a_n = \frac{\sin(n)}{n} \)

Look at data table. The values \( a_n \) seem to approach zero.

a. For \( \varepsilon = .3 \), need \( -0.3 < a_n < 0 + 0.3 \)
   
   \( \therefore -0.3 < a_n < 0.3 \)
   
   \( \text{From table, this holds if } n > 2 \)
   
   \( \therefore N(\varepsilon = .3) = 2. \)

b. \( \varepsilon = .05 \) need \( -0.05 < a_n < +0.05 \)

Seems like condition will hold if \( n > 17. \)

\( \therefore N(\varepsilon = .05) = 17. \)
<table>
<thead>
<tr>
<th>n</th>
<th>( an = \sin(n)/n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8415</td>
</tr>
<tr>
<td>2</td>
<td>0.4546</td>
</tr>
<tr>
<td>3</td>
<td>0.0470</td>
</tr>
<tr>
<td>4</td>
<td>-0.1892</td>
</tr>
<tr>
<td>5</td>
<td>-0.1918</td>
</tr>
<tr>
<td>6</td>
<td>-0.0466</td>
</tr>
<tr>
<td>7</td>
<td>0.0939</td>
</tr>
<tr>
<td>8</td>
<td>0.1237</td>
</tr>
<tr>
<td>9</td>
<td>0.0458</td>
</tr>
<tr>
<td>10</td>
<td>-0.0544</td>
</tr>
<tr>
<td>11</td>
<td>-0.0909</td>
</tr>
<tr>
<td>12</td>
<td>-0.0447</td>
</tr>
<tr>
<td>13</td>
<td>0.0323</td>
</tr>
<tr>
<td>14</td>
<td>0.0708</td>
</tr>
<tr>
<td>15</td>
<td>0.0434</td>
</tr>
<tr>
<td>16</td>
<td>-0.0180</td>
</tr>
<tr>
<td>17</td>
<td>-0.0566</td>
</tr>
<tr>
<td>18</td>
<td>-0.0417</td>
</tr>
<tr>
<td>19</td>
<td>0.0079</td>
</tr>
<tr>
<td>20</td>
<td>0.0456</td>
</tr>
<tr>
<td>21</td>
<td>0.0398</td>
</tr>
<tr>
<td>22</td>
<td>-0.0004</td>
</tr>
<tr>
<td>23</td>
<td>-0.0368</td>
</tr>
<tr>
<td>24</td>
<td>-0.0377</td>
</tr>
<tr>
<td>25</td>
<td>-0.0053</td>
</tr>
</tbody>
</table>
\[ S_4 \text{ (not S3)}: \quad S_n = 2 \cdot \frac{1}{n^2} \to 2 \]

Let \( \varepsilon = 0.1 \). Find \( N \) if \( n > N \) then \[ |S_n - 2| < 0.1 \]

\[ |(2 + \frac{1}{n^2}) - (2)| < 0.1 \]
\[ \left| \frac{1}{n^2} \right| < 0.1 \]
\[ \frac{1}{n^2} < 0.1 \]
\[ n^2 > \frac{1}{0.1} = 10 \]
\[ n > \sqrt{10} \approx 3.\ldots \]

Let \( N = \sqrt{10} \). If \( n > N \) then \[ |S_n - 2| < 0.1 \].
52. Equivalence Relation: Show that same cardinality satisfies transitivity, i.e.

if \( A \cong B \) (same cardinality) 
and \( B \cong C \) (same cardinality)

Then \( A \cong C \) (same cardinality)

We know there exist \( f: A \to B \) that is 1-1, onto
\( g: B \to C \) that is 1-1, onto

Required To Prove: That there is \( h: A \to C \) that is 1-1 and onto.

Define \( h: A \to C \) by composition:
\[ h(a) = g(f(a)) = g \circ f(a) \]
Claim: $h$ is onto.

pf: let $c \in C$. Then since $g : B \to C$ is onto, there exist $b \in B$ s.t. $g(b) = c$.

Now since $f : A \to B$ is onto, there exist $a \in A$ s.t. $f(a) = b$.

Now take this point $a \in A$.

$h(a) = g \circ f(a) = g(b) = c$

so $h$ is onto $C$: i.e., for any point $c \in C$, there is a point $a \in A$ s.t. $h(a) = c$.

Claim: $h$ is 1-1.

pf: suppose $h(a_1) = h(a_2)$

then $g \circ f(a_1) = g \circ f(a_2)$

so $g(b_1) = g(b_2)$ where $b_1 = f(a_1)$ and $b_2 = f(a_2)$.

Since we know $g$ is 1-1, we conclude that $b_1 = b_2$.

Hence $f(a_1) = f(a_2)$.

Since $f$ is 1-1, we conclude $a_1 = a_2$.

so $h$ is 1-1.
S_2:

a) increasing, bounded, converges to 3
(no proof needed)

\[ s_n = -3 - \frac{1}{n} \]

\[ s_1 = -4, \quad s_2 = -3\frac{1}{2}, \quad s_3 = -3\frac{1}{3}, \quad s_4 = -3\frac{1}{4}, \ldots \]

b) oscillating, bounded, converges to +4

\[ s_n = 4 + \frac{(-1)^n}{n} \]

\[ s_1 = 4 + 1 = 5, \quad s_2 = 4 + \frac{1}{2} = 4\frac{1}{2}, \quad s_3 = 4 - \frac{1}{3} = 3\frac{2}{3}, \quad s_4 = 4\frac{1}{4}, \ldots \]

c) oscillating that diverges

\[ s_n = (-1)^n \]

\[ s_1 = -1, \quad s_2 = 2, \quad s_3 = -3, \quad s_4 = 4, \ldots \]

d) increasing, unbounded that diverges

\[ s_n = n^2 \]

\[ s_1 = 1, \quad s_2 = 4, \quad s_3 = 9, \ldots \]

e) decreasing, bounded, converges to 3

\[ s_n = 3 + \frac{1}{n} \]

\[ s_1 = 4, \quad s_2 = 3\frac{1}{2}, \quad s_3 = 3\frac{1}{3}, \ldots \]
S 3 a. \( s_n = 2 + \frac{1}{n^2} \). \( s_n \to 2 \)

**Proof:** want \( |s_n - 2| < \varepsilon \)

\[ \iff |2 + \frac{1}{n^2} - 2| < \varepsilon \]

\[ \iff \left| \frac{1}{n^2} \right| < \varepsilon \]

\[ \iff \frac{1}{n^2} < \varepsilon \]

\[ \iff n^2 > \frac{1}{\varepsilon} \]

\[ \iff n > \frac{1}{\sqrt{\varepsilon}} \]

*Not required to write it this way.*

Let \( \varepsilon > 0 \). Choose \( N = N(\varepsilon) = \frac{1}{\sqrt{\varepsilon}} \).

If \( n > N = \frac{1}{\sqrt{\varepsilon}} \) then \( |s_n - 2| < \varepsilon \)

by above calculation.

So \( s_n \to 2 \) by definition of limit.
\[ S_n = 3 + 2^n \quad \Rightarrow \quad 3 = 1 \]

**Proof:** want \( |S_n - L| < \varepsilon \)

\[ |3 + \frac{1}{2^n} - 3| < \varepsilon \]

\[ \Rightarrow \quad \left| \frac{1}{2^n} \right| < \varepsilon \]

\[ \Rightarrow \quad \frac{1}{2^n} < \varepsilon \quad \text{since} \quad \frac{1}{2^n} > 0 \]

\[ \Rightarrow \quad 2^n > \frac{1}{\varepsilon} \]

\[ \Rightarrow \quad \log(2^n) > \log\left(\frac{1}{\varepsilon}\right) \]

\[ \Rightarrow \quad n \ln 2 > \ln \varepsilon - \ln 1 \]

\[ \Rightarrow \quad n > \frac{-\ln \varepsilon}{\ln 2} \]

**Note:** 

\[ y = \ln x \]

If \( \varepsilon < 1 \), then \( \ln(\varepsilon) < 0 \)

\[ \Rightarrow -\ln(\varepsilon) > 0 \]

So \( n > \) (positive number)

Given \( \varepsilon > 0 \), let \( N = \frac{-\ln(\varepsilon)}{\ln 2} \)

If \( n > N \), then the above calculation shows that \( |S_n - L| < \varepsilon \).
54. a) $S_n = \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]$ for $n \geq 2$

\[
S_2 = \left[ \frac{1}{2}, \frac{1}{2} \right]
\]
\[
S_3 = \left[ \frac{1}{3}, \frac{2}{3} \right]
\]
\[
S_4 = \left[ \frac{1}{4}, \frac{4}{4} \right]
\]
\[
S_n = \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]
\]

$\bigcup_{n=2}^{\infty} S_n = \{ x \in \mathbb{R} : x \in S_k \text{ for some } k \}$

$= (0, 1)$.

Note: $0 \notin \bigcup_{n=2}^{\infty} S_n$ since $0$ never in any of these.

$b) ~ T_n = (1 - \frac{1}{n}, 1 + \frac{1}{n})$ [mean $\left( 1 - \frac{1}{n}, 2 + \frac{1}{n} \right)$-ops]

\[
T_2 = \left( \frac{1}{2}, \frac{3}{2} \right)
\]
\[
T_3 = \left( \frac{2}{3}, \frac{4}{3} \right)
\]
\[
T_4 = \left( \frac{4}{5}, \frac{6}{5} \right)
\]
\[ \bigwedge_{n \geq 2} T_n = \mathbb{S}_1. \]

1 is in each set.
No other point is in every \( T_n \) set.